High-order short-time expansions for ATM option prices under the CGMY model

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Abstract

The short-time asymptotic behavior of option prices for a variety of models with jumps has received much attention in recent years. In the present work, a novel second-order approximation for ATM option prices under the CGMY Lévy model is derived and, then, extended to a model with an additional independent Brownian component. Our method of proof is based on an integral representation of the option price involving the tail probability of the log-return process under the share measure and a suitable change of probability measure under which the process becomes stable. This approach is sufficiently efficient to produce the third-order asymptotic behavior of the option prices and, moreover, is expected to apply to many other popular classes of Lévy processes which satisfy the fundamental property of being stable under a suitable change of probability measure. Our results shed new light on the connection between both the volatility of the continuous component and the jump parameters and the behavior of ATM option prices near expiration. In the case of an additional Brownian component, the second-order term, in time-t, is of the form $d_2\, t^{(3-Y)/2}$, with the coefficient d_2 depending only on the overall jump intensity parameter C and the tail-heaviness parameter Y. This extends the known result that the leading term is $(\sigma/\sqrt{2\pi})t^{1/2}$, where σ is the volatility of the continuous component. In contrast, under a pure-jump CGMY model, the dependence on the two parameters C and Y is already reflected in the leading term, which is of the form $d_1 t^{1/Y}$. Information on the relative frequency of negative and positive jumps appears only in the second-order term, which is shown to be of the form $d_2 t$ and whose order of decay turns out to be independent of Y. The third-order asymptotic behavior of the option prices as well as the asymptotic behavior of the corresponding Black-Scholes implied volatilities are also addressed. Our numerical results show that first-order term typically exhibits extremely poor performance and that the second-order term significantly improves the approximation's accuracy.

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1 Introduction

It is generally recognized that the standard option pricing model of Black-Scholes is inconsistent with options data, while remaining a widely used model in practice because of its simplicity. Exponential Lévy models generalize the classical Black-Scholes setup by allowing jumps in stock prices while preserving the independence and stationarity of returns. There are several reasons for introducing jumps in financial modeling. First of all, asset prices do jump, and some risks simply cannot be handled within continuous-paths models. Second, historical asset prices exhibit distributions with so-called stylized features, such as heavy tails, high kurtosis, volatility clustering and leverage effects, which are hard to replicate within purely-continuous frameworks. Finally, market prices of vanilla options exhibit skewed implied volatilities (relative to changes in the strikes), in contrast to the classical Black-Scholes model which predicts a flat implied volatility smile. Moreover, the fact that the implied volatiliy smile and skewness phenomenon becomes much more pronounced for short maturities is a clear indication of the presence of jumps.

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One of the first applications of jump processes in financial modeling is due to Mandelbrot [26], who suggested a pure-jump stable Lévy process Z to model power-like tails and self-similar behavior in cotton price returns. Merton [28] and Press [31] subsequently considered option pricing and hedging problems under an exponential compound Poisson process with Gaussian jumps and an additive independent non-zero Brownian component. A similar exponential compound Poisson jump-diffusion model was more recently studied in Kou [22], where the jump sizes are distributed according to an asymmetric Laplace law. For infinite activity exponential Lévy models, Barndorff-Nielsen [1] introduced the normal inverse Gaussian (NIG) model, while the extension to the generalized hyperbolic class was studied by Eberlein, Keller and Prause [10]. Madan and Seneta [25] introduced the symmetric variance gamma (VG) model while its asymmetric extension was later studied by Madan and Milne [24] and Madan, Carr and Chang [23]. Both models are built on Brownian subordination; the main difference being that the log-return process in the NIG model is an infinite variation process with stable like ($\alpha = 1$) behavior of small jumps, while in the VG model, the log-price is of finite variation with infinite but relatively low activity of small jumps. The class of "tempered stable" processes was first introduced by Koponen [21] and further developed by Carr, Geman, Madan and Yor [5], who introduced the terminology CGMY. The CGMY model is a particular case of the more general KoBoL class of [4] and was also previously proposed for financial modeling in [7] and [27]. Nowadays, the CGMY model is considered to be a prototype of the general class of models with jumps and enjoys widespread applicability.

Stemming in part from its importance for model calibration and testing, small-time asymptotics of option prices have received a lot of attention in recent years (see, e.g., [2], [3], [11], [12], [13], [16], [17], [18], [19], [20], [30], [32], [38]). We shall review here only the studies most closely related to ours, focusing in particular on the at-the-money (ATM) case. Carr and Wu [9] first analyzed, partially via heuristic arguments, the first order asymptotic behavior of an Itô semimartingale with jumps. Concretely, ATM option prices of pure-jump models of bounded variation decrease at the rate $O(t)$, while they are just $O(\sqrt{t})$ under the presence of a Brownian component. By considering a stable pure-jump component, [9] also showed that, in general, the rate could be $O(t^{\beta})$, for some $\beta \in (0,1)$. Muhle-Karbe and Nutz [29] formally showed that, under the presence of a continuous-time component, the leading term of ATM option prices is of order $\sqrt{t}$, for a relatively general class of Itô models, while for a more general type of Itô processes with α-stable-like small jumps, the leading term is $O(t^{1/\alpha})$ (see also [13, Proposition 4.2], [15, Theorem 3.7], and [38, Proposition 5] for related results in exponential Lévy models). However, none of the these papers obtained second or higher order asymptotics for the ATM option prices, which are arguably more relevant for calibration purposes, given that the most liquid options are of this type.

In the present paper, we study the small-time behavior for at-the-money (ATM) call (or equivalently, put) option prices

$$\mathbb{E}\left(S_t - S_0\right)^+ = S_0 \mathbb{E}\left(e^{X_t} - 1\right)^+, \tag{1.1}$$

under the exponential Lévy model

$$S_t := S_0 e^{X_t}, \tag{1.2}$$

where X is the superposition of a CGMY Lévy process $(L_t)_{t\geq 0}$ and of an independent Brownian motion $(\sigma W_t)_{t\geq 0}$; i.e.,

$$X_t = L_t + \sigma W_t, \tag{1.3}$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion independent of L. Here, as usual, x^+ is the positive part of x. The first order asymptotic behavior of (1.1) in short-time under the model (1.3) takes the form:

$$\lim_{t\to 0} t^{-1/Y} \mathbb{E}(S_t - S_0)^+ = S_0 \mathbb{E}(Z^+), \tag{1.4}$$

where Z is a symmetric stable random variable with $\alpha = Y$ under $\mathbb{P}$. When $\sigma \neq 0$, $Z \sim \mathcal{N}(0, \sigma^2)$ ($\alpha = 2$) and, thus, $\mathbb{E}(Z^+) = \sigma/\sqrt{2\pi}$ (see [38] and [32]). When $\sigma = 0$ and $\alpha = Y$, the characteristic function of Z is explicitly given (see [13] and [38]) by

$$\mathbb{E}e^{iuZ} = e^{-2C\Gamma(-Y)|\cos(\frac{1}{2}Y\pi)|\,|u|^Y}.$$

In that case, (see (25.6) in [37]),

$$d_1 := \mathbb{E}(Z^+) = \frac{1}{\pi}\Gamma\left(1 - \frac{1}{Y}\right)\left(2C\Gamma(-Y)\left|\cos\left(\frac{\pi Y}{2}\right)\right|\right)^{1/Y}. \tag{1.5}$$

Interestingly enough, under the presence of a continuous component, the first-order asymptotic term only reflects information on the continuous-time volatility, in sharp contrast with the pure-jump case where the leading term depends on the overall jumps-intensity parameter C and the index Y, which in turn controls the tail-heaviness of the distributions.

Below, we also obtain a second order correction term for the approximation (1.4). The derivation of the second-order results builds on two facts. First, as in [13], we make use of the following representation of Carr and Madan [6]:

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0)^+ = \mathbb{P}^*(X_t > E) = \int_0^\infty e^{-x}\mathbb{P}^*(X_t > x)dx\,, \tag{1.6}$$

where $\mathbb{P}^*$ is the martingale probability measure obtained when one takes the stock as the numéraire (i.e., $\mathbb{P}^*(A) := \mathbb{E}\left(S_t 1_A\right)$) and E is an independent mean-one exponential random variable under $\mathbb{P}^*$. The measure $\mathbb{P}^*$ is sometimes called the share measure (see [6]). Notice that under $\mathbb{P}^*$, $(X_t)_{t\geq 0}$ also admits a decomposition similar to (1.3),

$$X_t = L_t^* + \sigma W_t^*, \quad t \geq 0, \tag{1.7}$$

where $W^* := (W_t^*)_{t\geq 0}$ is a Wiener process and $L^* := (L_t^*)_{t\geq 0}$ is also a CGMY process, independent of W^*. Second, we change probability measures from $\mathbb{P}^*$ to a probability measure $\widetilde{\mathbb{P}}$, under which $(L_t^*)_{t\geq 0}$ is a stable Lévy process and $(W_t^*)_{t\geq 0}$ is still a standard Brownian motion independent of L^*. We show that the second-order asymptotic behavior of the ATM call option price (1.1) in short-time is then of the form

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0)^+ = d_1 t^{\frac{1}{Y}} + d_2 t + o(t), \qquad (t \to 0),$$

in the pure-jump CGMY case ($\sigma = 0$), while in the case of a non-zero independent Brownian component ($\sigma \neq 0$),

$$\frac{1}{S_0}\mathbb{E}(S_t - S_0)^+ = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + o\left(t^{\frac{3-Y}{2}}\right), \qquad (t \to 0),$$

for different constants d_1 and d_2 that we will determine explicitly. To wit, we found that, under the presence of a nonzero Gaussian component, the second-order term depends only on the overall jump intensity parameter C and the tail-heaviness parameter Y. The parameters G and M (which control the relative frequency of negative and positive jumps) do not appear until the next order term. However, for a pure-jump case, the parameters G and M are already present in the second-order term. The above asymptotic behaviors should also be compared to the corresponding behavior under the standard Black-Scholes model, where it is known that (see, e.g., [18, Corollary 3.4])

$$\mathbb{E}(e^{\sigma W_t - \frac{\sigma^2}{2}t} - 1)^+ = \frac{\sigma}{\sqrt{2\pi}}t^{\frac{1}{2}} - \frac{\sigma^3}{24\sqrt{2\pi}}t^{\frac{3}{2}} + O(t^{\frac{5}{2}}).$$

Our method of proof is sharp enough to produce the third-order asymptotic behavior of the option prices (see Remark 3.4 and 4.4 below) and, moreover, is expected to apply to other popular classes of Lévy processes, which satisfy the fundamental property of being stable under a suitable change of probability measure such as tempered stable processes in the sense of Rosiński [35] (this will be presented elsewhere). Finally, the asymptotic behavior of the corresponding Black-Scholes implied volatilities are also addressed.

The present paper is organized as follows. Section 2 contains preliminary results on the CGMY model, some probability measure transformations, and asymptotic results for stable Lévy processes which will be needed throughout the paper. Section 3 establishes the second-order asymptotics of the call option price under the pure-jump CGMY model ($\sigma = 0$). Section 4 establishes the second-order asymptotics of the call-option price under the CGMY model with an additional independent non-zero Brownian component ($\sigma \neq 0$). In Section 5, we assess the performance of our asymptotic expansions through a detailed numerical analysis. The proofs of our main results are deferred to the Appendices.

2 Setup and preliminary results

2.1 The CGMY model

Throughout, $(L_t)_{t\geq 0}$ stands for a CGMY Lévy process defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with corresponding parameters $C, G, M > 0$ and $Y \in (1,2)$. That is, L is a pure-jump Lévy process with characteristic function

$$\mathbb{E}\left(e^{iuL_t}\right) = \exp\left(t\left[icu + C\Gamma(-Y)\left((M-iu)^Y + (G+iu)^Y - M^Y - G^Y\right)\right]\right). \tag{2.1}$$

Let $X_t = \sigma W_t + L_t,\ t \geq 0$, where $(W_t)_{t\geq 0}$ is a standard Brownian motion, independent of $(L_t)_{t\geq 0}$, defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We call the process X the (generalized) CGMY model.

We assume zero interest rate and that $\mathbb{P}$ is a martingale measure for the exponential Lévy model $S_t = S_0 e^{X_t}$. In particular, $M > 1$ and the characteristic function φ_t of X_t is given by

$$\varphi_t(u) = \mathbb{E}\left(e^{iuX_t}\right) = \exp\left(t\left[icu - \frac{\sigma^2 u^2}{2} + C\Gamma(-Y)\left((M-iu)^Y + (G+iu)^Y - M^Y - G^Y\right)\right]\right), \tag{2.2}$$

with

$$c = -C\Gamma(-Y)\left((M-1)^Y + (G+1)^Y - M^Y - G^Y\right) - \frac{\sigma^2}{2}; \tag{2.3}$$

see, e.g., Proposition 4.2 in [38]. In particular, we note that $\gamma := \mathbb{E}X_1 = \mathbb{E}L_1$ is given by

$$\gamma = c - CY\Gamma(-Y)(M^{Y-1} - G^{Y-1}). \tag{2.4}$$

The Lévy triplet of $(X_t)_{t\geq 0}$ (relative to the truncation function $x\mathbf{1}_{\{|x|\leq 1\}}$) is denoted by (b, σ^2, ν). Thus, ν and b are given by

$$\nu(dx) = \left(\frac{Ce^{-Mx}}{x^{1+Y}}\,\mathbf{1}_{\{x>0\}} + \frac{Ce^{Gx}}{|x|^{1+Y}}\,\mathbf{1}_{\{x<0\}}\right)dx, \tag{2.5}$$

$$b = c - \int_{|x|>1} x\nu(dx) - CY\Gamma(-Y)(M^{Y-1} - G^{Y-1}). \tag{2.6}$$

Without loss of generality, we also assume throughout that $(X_t)_{t\geq 0}$ is the canonical process $X_t(\omega) = \omega(t)$ defined on the canonical space $\Omega = \mathbb{D}([0,\infty), \mathbb{R})$ (the space of càdlàg functions $\omega : [0,\infty) \to \mathbb{R}$) equipped with the σ-field $\mathcal{F} = \sigma(X_s : s \geq 0)$ and the right-continuous filtration $\mathcal{F}_t := \cap_{s>t}\sigma(X_u : u \leq s)$.

2.2 Probability measure transformations

Following a density transformation construction as given in Sato [37] (see Definition 33.4 and Example 33.4 there) and using the martingale condition $\mathbb{E}e^{X_t} = 1$, we define a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ such that

$$\frac{d\mathbb{P}^*|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = e^{X_t}, \qquad (t \geq 0); \tag{2.7}$$

i.e., $\mathbb{P}^*(B) = \mathbb{E}\left(e^{X_t}\mathbf{1}_B\right)$, for any $B \in \mathcal{F}_t$ and $t \geq 0$. The measure $\mathbb{P}^*$ can be interpreted as the martingale measure when using the stock price as the numéraire. Under $\mathbb{P}^*$, $(X_t)_{t\geq 0}$ is also a Lévy process and its characteristic function is given by

$$\mathbb{E}^*(e^{iuX_t}) = \exp\left(t\left[ic^*u - \frac{\sigma^2u^2}{2} + C\Gamma(-Y)\left((M^*-iu)^Y + (G^*+iu)^Y - {M^*}^Y - {G^*}^Y\right)\right]\right), \tag{2.8}$$

with (see Appendix C)

$$M^* = M - 1, \quad G^* = G + 1, \quad c^* = c + \sigma^2.$$

It is clear from (2.8) that, under $\mathbb{P}^*$, $(X_t)_{t\geq 0}$ can also be decomposed as in (1.7), where $(W_t^*)_{t\geq 0}$ is again a Wiener process while $(L_t^*)_{t\geq 0}$ is still a CGMY process, independent of W^*, but with parameters C, Y, $M = M^*$, and $G = G^*$. Hereafter, we denote the Lévy triplet of $(X_t)_{t\geq 0}$ under $\mathbb{P}^*$ by $(b^*, (\sigma^*)^2, \nu^*)$, where $\sigma^* = \sigma$, $\nu^*(dx) = e^x \nu(dx)$, and

$$b^* := c^* - \int_{|x|>1} x\nu^*(dx) - CY\Gamma(-Y)((M^*)^{Y-1} - (G^*)^{Y-1}). \tag{2.9}$$

As explained in the introduction, an important tool in the sequel is to change the probability measures from $\mathbb{P}^*$ to a probability measure $\widetilde{\mathbb{P}}$, under which $(L_t^*)_{t\geq 0}$ is a stable Lévy process and $(W_t^*)_{t\geq 0}$ is still a Wiener process independent of L^*. Concretely, let

$$\tilde{\nu}(dx) := C|x|^{-Y-1}dx, \qquad \tilde{b} = b^* + \int_{|x|\leq 1} x(\tilde{\nu} - \nu^*)(dx).$$

Note that $\tilde{\nu}$ is the Lévy measure of a symmetric stable Lévy process and, also,

$$\tilde{\nu}(dx) = e^{\varphi(x)}\nu^*(dx),$$

with

$$\varphi(x) := M^* x\, \mathbf{1}_{\{x>0\}} - G^* x\, \mathbf{1}_{\{x<0\}}.$$

Hence, by virtue of Theorem 33.1 in [37], there exists a probability measure $\widetilde{\mathbb{P}}$ locally equivalent[1] to $\mathbb{P}^*$ such that $(X_t)_{t\geq 0}$ is a Lévy process with Lévy triplet $(\tilde{b}, \sigma^2, \tilde{\nu})$ under $\widetilde{\mathbb{P}}$. Throughout, $\widetilde{\mathbb{E}}$ denotes the expectation under $\widetilde{\mathbb{P}}$.

In light of (2.9) and since $\widetilde{\mathbb{E}}X_1 = \widetilde{\mathbb{E}}L_1^* = \tilde{b} + \int_{\{|x|>1\}} x\tilde{\nu}(dx)$, it can be shown (see Appendix C) that

$$\tilde{\gamma} := \widetilde{\mathbb{E}}X_1 = -C\Gamma(-Y)\left((M-1)^Y + (G+1)^Y - M^Y - G^Y\right) + \frac{\sigma^2}{2}. \tag{2.10}$$

Next, we recall that the centered process $(Z_t)_{t\geq 0}$, given by

$$Z_t := L_t^* - t\tilde{\gamma}, \tag{2.11}$$

is symmetric and strictly Y-stable[2] under $\widetilde{\mathbb{P}}$ and, thus, is self-similar; i.e.,

$$(t^{-1/Y} Z_{ut})_{u\geq 0} \stackrel{\mathfrak{D}}{=} (Z_u)_{u\geq 0}, \tag{2.12}$$

for any $t > 0$. We also need the following representation of the density process (see Theorem 33.2 in [37]):

$$\frac{d\widetilde{\mathbb{P}}\big|_{\mathcal{F}_t}}{d\mathbb{P}^*\big|_{\mathcal{F}_t}} = e^{U_t}, \tag{2.13}$$

with

$$U_t := \lim_{\epsilon\to 0}\left(\sum_{s\leq t: |\Delta X_s|>\varepsilon} \varphi(\Delta X_s) - t\int_{|x|>\varepsilon} (e^{\varphi(x)} - 1)\nu^*(dx)\right).$$

The process $(U_t)_{t\geq 0}$ can be expressed in terms of the jump-measure $N(dt, dx) := \#\{(s, \Delta X_s) \in dt \times dx\}$ of the process $(X_t)_{t\geq 0}$ and its compensator $\bar{N}(dt, dx) := N(dt, dx) - \tilde{\nu}(dx)dt$ (under $\widetilde{\mathbb{P}}$); namely,

$$U_t = M^* \bar{U}_t^+ - G^* \bar{U}_t^- + \eta t, \tag{2.14}$$

[1] Equivalently, there exists a process $(U_t)_t$ such that $\widetilde{\mathbb{P}}(B) = \mathbb{E}^*(e^{U_t}\mathbf{1}_B)$, for $t \geq 0$ and $B \in \mathcal{F}_t$.
[2] Concretely, its scale, skewness, and location parameters are $2C\Gamma(-Y)|\cos(\pi Y/2)|$, 0, and 0, respectively.

where

$$\bar{U}_t^+ := \int_0^t \int_{(0,\infty)} x\bar{N}(ds,dx), \quad \bar{U}_t^- := \int_0^t \int_{(-\infty,0)} x\bar{N}(ds,dx), \tag{2.15}$$

$$\begin{aligned}\eta &:= C\int_{0^+}^{\infty}(e^{-M^*x}-1+M^*x)x^{-Y-1}dx + C\int_{-\infty}^{0^-}(e^{G^*x}-1-G^*x)|x|^{-Y-1}dx \\ &= C\Gamma(-Y)\left((M^*)^Y+(G^*)^Y\right),\end{aligned} \tag{2.16}$$

where in the last equality we used the analytic continuation presented in [37] (see (14.19) therein). Finally, let us also note the following decomposition of the process X in terms of the previously defined processes:

$$X_t = Z_t + t\tilde{\gamma} + \sigma W_t^* = \bar{U}_t^+ + \bar{U}_t^- + t\tilde{\gamma} + \sigma W_t^*. \tag{2.17}$$

The following table summarizes the different probability measures used in this paper:

Prob. Measure	$(L_t)_t$ distribution	Density wrt $\mathbb{P}$
$\mathbb{P}$	CGMY(C,G,M,Y)	1
$\mathbb{P}^*$	CGMY(C,G^*,M^*,Y)	e^{X_t}
$\widetilde{\mathbb{P}}$	Stable$(\beta=0;\alpha=Y)$	$e^{X_t+U_t}$

2.3 Some needed properties of stable Lévy processes

Let us now collect some well-known results on stable Lévy processes needed in the sequel. First, from (2.15), it is clear that $(\bar{U}_t^+)_{t\geq 0}$ and $(-\bar{U}_t^-)_{t\geq 0}$ are independent and identically distributed one-sided Y-stable processes[3] under $\widetilde{\mathbb{P}}$. Hence, the common transition density of $\bar{U}_t^+$ and $-\bar{U}_t^-$, denoted by $p(t,x)$, exists (cf. [37, Proposition 2.5]). Moreover, the following result for the asymptotic behavior of the transition density is known (see, e.g., [34] and [14]):

$$\lim_{t\to 0}\frac{1}{t}p(t,u) = s(u), \qquad (u\neq 0),$$

where s is the Lévy density of the Lévy process $(\bar{U}_t^+)_{t\geq 0}$. In particular, since by construction the Lévy measure of $(\bar{U}_t^+)_{t\geq 0}$ is $\tilde{\nu}_+(du) = Cu^{-Y-1}\mathbf{1}_{\{u>0\}}du$, the Lévy density s is just $Cu^{-Y-1}\mathbf{1}_{\{u>0\}}$, $u\neq 0$, and we get:

$$\lim_{t\to 0}\frac{1}{t}p(t,u) = Cu^{-Y-1}\mathbf{1}_{\{u>0\}}, \quad u\neq 0. \tag{2.18}$$

Equivalently, by the self-similarity of $(\bar{U}_t^+)_{t\geq 0}$, we have

$$p(t,u) = t^{-1/Y}p(1,t^{-1/Y}u),$$

and (2.18) can be casted as follows by setting $x = t^{-1/Y}u$:

$$p(1,x) \sim Cx^{-Y-1}, \qquad (x\to\infty). \tag{2.19}$$

As a consequence,

$$\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq x\right) = \widetilde{\mathbb{P}}\left(-\bar{U}_1^- \geq x\right) \sim \frac{C}{Y}x^{-Y}, \tag{2.20}$$

[3] Concretely, its scale, skewness, and location parameters are $C|\cos(\pi Y/2)|\Gamma(-Y)$, 1, and 0, respectively.

as $x \to \infty$. Equivalently, plugging $x = t^{-1/Y}v$ and using the self-similarity of $\bar{U}^+$ and $\bar{U}^-$, we recover the well-known result:

$$\lim_{t\to 0} \frac{1}{t}\widetilde{\mathbb{P}}\Big(\pm \bar{U}_t^{\pm} \geq v\Big) = \tilde{\nu}([v,\infty)) = \frac{C}{Y}v^{-Y}. \tag{2.21}$$

In particular, there exists $N > 0$, such that for all $0 < t \leq 1$ and $v > 0$ satisfying $t^{-1/Y}v > N$,

$$\widetilde{\mathbb{P}}(\bar{U}_1^+ \geq t^{-1/Y}v) \leq \frac{2C}{Y}tv^{-Y}, \qquad \widetilde{\mathbb{P}}(-\bar{U}_1^- \geq t^{-1/Y}v) \leq \frac{2C}{Y}tv^{-Y}. \tag{2.22}$$

The following result sharpens (2.22). Its proof is presented in Appendix C.

Lemma 2.1. *There exists a constant* $0 < \kappa < \infty$ *such that for* any $0 < t \leq 1$ *and* $v > 0$*,*

$$\widetilde{\mathbb{P}}(\bar{U}_1^+ \geq t^{-1/Y}v) \leq \kappa tv^{-Y}, \qquad \widetilde{\mathbb{P}}(-\bar{U}_1^- \geq t^{-1/Y}v) \leq \kappa tv^{-Y}.$$

Therefore, since $Z_1 = \bar{U}_1^+ - \bar{U}_1^-$,

$$\widetilde{\mathbb{P}}(Z_1 \geq t^{-1/Y}v) = \widetilde{\mathbb{P}}(Z_t \geq v) \leq 2^{Y+1}\kappa tv^{-Y} \leq 8\kappa tv^{-Y}, \tag{2.23}$$

for any $0 < t \leq 1$ and $v > 0$. Note also that

$$\widetilde{\mathbb{P}}(Z_t \geq v) = \widetilde{\mathbb{P}}(Z_1 \geq t^{-1/Y}v) \sim t\tilde{\nu}([v,\infty)) = t\,\frac{C}{Y}v^{-Y}, \quad (t \to 0), \tag{2.24}$$

and that the probability density p_Z of Z_1 is such that

$$p_Z(v) \sim Cv^{-Y-1}, \qquad (v \to \infty). \tag{2.25}$$

The following identity for $\widetilde{U}_t := M^*\bar{U}_t^+ - G^*\bar{U}_t^-$ will also be needed in sequel:

$$\widetilde{\mathbb{E}}\Big(e^{-t^{1/Y}\widetilde{U}_1}\Big) = \mathbb{E}^*\Big(e^{-t^{1/Y}M^*\bar{U}_1^+}\Big)\mathbb{E}^*\Big(e^{t^{1/Y}G^*\bar{U}_1^-}\Big) = \exp(\eta t). \tag{2.26}$$

The relation above follows from the representations (2.15), the independence of $\bar{U}^+$ and $\bar{U}^-$, and the form of the characteristic function of a Poisson integral.

3 The pure-jump CGMY model

In this section, we find the second-order asymptotic behavior for the at-the-money call option prices (1.1) in the pure-jump CGMY model. The proofs of all results in the section are deferred to the Appendix A. Throughout this section, $(X_t)_{t\geq 0}$ is a Lévy process with triplet $(b, 0, \nu)$ as introduced in Section 2. As explained in the introduction, the first order asymptotic behavior is given by (1.4). Before stating our first result, we need to rewrite the call option price (1.1) in a suitable form.

Lemma 3.1. *In terms of the probability measure* $\widetilde{\mathbb{P}}$ *defined in (2.13) and the parameter* $\tilde{\gamma}$ *defined in (2.10), we have that*

$$t^{-\frac{1}{Y}}\frac{1}{S_0}\mathbb{E}(S_t - S_0)^+ = e^{-(\tilde{\gamma}+\eta)t}\int_{-\tilde{\gamma}t^{1-1/Y}}^{\infty} e^{-t^{1/Y}v}\,\widetilde{\mathbb{E}}\left(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1\geq v\}}\right)dv, \tag{3.1}$$

where $\widetilde{U}_t := M^*\bar{U}_t^+ - G^*\bar{U}_t^-$*, and* $(\bar{U}_t^+)_{t\geq 0}$ *and* $(\bar{U}_t^-)_{t\geq 0}$ *are defined as in (2.15).*

The following result gives the second-order asymptotic behavior of at-the-money call option prices under the pure-jump CGMY model.

Theorem 3.2. *Under the exponential CGMY model (1.2) without Brownian component,*

$$\lim_{t\to 0} t^{\frac{1}{Y}-1}\left(t^{-\frac{1}{Y}}\frac{1}{S_0}\mathbb{E}(S_t-S_0)^+ - \widetilde{\mathbb{E}}(Z_1^+)\right) = \vartheta+\eta+\frac{\tilde{\gamma}}{2}, \tag{3.2}$$

where η and $\tilde{\gamma}$ are respectively given as in (2.16) and (2.10), and

$$\vartheta := -C\Gamma(-Y)\Big((M^*+1)^Y + (G^*)^Y\Big). \tag{3.3}$$

Remark 3.3. Using (3.3), (2.16), and (2.10), it turns out that

$$d_2 := \vartheta+\eta+\frac{\tilde{\gamma}}{2} = \frac{C\Gamma(-Y)}{2}\left((M-1)^Y - M^Y - (G+1)^Y + G^Y\right).$$

Hence, the second-order asymptotic behavior of the ATM call option price (1.1) in short-time is

$$\frac{1}{S_0}\mathbb{E}(S_t-S_0)^+ = d_1 t^{\frac{1}{Y}} + d_2 t + o(t), \qquad (t\to 0), \tag{3.4}$$

with $d_1 = \widetilde{\mathbb{E}}(Z_1^+)$ given as in (1.5):

$$d_1 = \frac{1}{\pi}\Gamma\left(1-\frac{1}{Y}\right)\left(2C\Gamma(-Y)\left|\cos\left(\frac{\pi Y}{2}\right)\right|\right)^{\frac{1}{Y}}.$$

Broadly speaking, the first-order term synthesizes only the information on the tail-heaviness index Y and the overall jump-intensity parameter C, while the second-order term incorporates also the information on the relative intensities of negative and positive jumps (controlled by the parameters G and M). Note also that $d_2 < -C\Gamma(-Y) \le -2C$.

Remark 3.4. The proof of Theorem 3.2 (see Appendix A) also provides the higher order asymptotics of the ATM call option price under the pure jump CGMY model. Indeed, it is clear that the term D_2 defined in (A.5) is such that $D_2(t) \sim -\eta\widetilde{\mathbb{E}}(Z_1^+)t$. Moreover, the second-order term of $D_1(t)$ therein can be shown to be $O(t^{2-\frac{1}{Y}})$, while $D_3(t) = o(D_2(t))$, as $t\to 0$. Therefore, as $t\to 0$, and since $1<Y<2$,

$$\frac{1}{S_0}\mathbb{E}(S_t-S_0)^+ = d_1 t^{\frac{1}{Y}} + d_2 t - \eta\widetilde{\mathbb{E}}(Z_1^+)t^{1+\frac{1}{Y}} + o(t^{1+\frac{1}{Y}}). \tag{3.5}$$

Let $\hat{\sigma}(t)$ denote the ATM Black-Scholes implied volatility at maturity t with zero interest rates. The following result gives the asymptotic behavior of $\hat{\sigma}(t)$ as $t\to 0$.

Proposition 3.5. *Under the exponential CGMY model (1.2) without Brownian component, the implied volatility $\hat{\sigma}(t)$ has the following small-time behavior:*

$$\hat{\sigma}(t) = \sigma_1 t^{\frac{1}{Y}-\frac{1}{2}} + \sigma_2 t^{\frac{1}{2}} + o(t^{\frac{1}{2}}), \qquad t\to 0, \tag{3.6}$$

where

$$\sigma_1 := \sqrt{2\pi}\,\widetilde{\mathbb{E}}(Z_1^+), \tag{3.7}$$

$$\sigma_2 := \sqrt{\frac{\pi}{2}}C\Gamma(-Y)\left((M-1)^Y - M^Y - (G+1)^Y + G^Y\right). \tag{3.8}$$

4 The CGMY model with Brownian component

In this part, we consider the CGMY model with non-zero Brownian component. Concretely, throughout, $(X_t)_{t\ge 0}$ is a Lévy process with triplet (b,σ^2,ν) as introduced in Section 2 and $\sigma\neq 0$. In that case, it follows from (2.2) that

$$\lim_{t\to 0}\mathbb{E}^*\left(\exp(iuX_t/\sqrt{t})\right) = \exp\left(-\frac{1}{2}\sigma^2u^2\right),$$

and, thus, $(X_t/\sqrt{t})$ converges weakly to the centered Gaussian distribution with variance σ^2. Equivalently, recalling that under $\mathbb{P}^*$, $(W_t^*)_{t\geq 0}$ is a standard Brownian motion, it follows that

$$\lim_{t\to 0} \mathbb{P}^*(X_t/\sqrt{t} \geq x) = \mathbb{P}^*(\sigma W_1^* \geq x). \tag{4.1}$$

The first order asymptotic behavior for the ATM European call options in this mixed model was obtained in [38] using Fourier methods. We present, in Appendix B, a probabilistic proof based on (4.1) and following an approach similar to that in [13].

Proposition 4.1. *In the setting of Section 2, the at-the-money European call option price has the following asymptotic behavior:*

$$\lim_{t\to 0} t^{-1/2}\mathbb{E}(S_t - S_0)_+ = S_0\sigma\mathbb{E}^*(W_1^*)_+. \tag{4.2}$$

Next, we give the second-order correction term for the at-the-money European call option price. As before, we change the probability measure $\mathbb{P}^*$ to $\widetilde{\mathbb{P}}$ so that $X_t = Z_t + t\tilde{\gamma} + \sigma W_t^*$, with $(Z_t)_{t\geq 0}$ a symmetric strictly Y-stable Lévy process under $\widetilde{\mathbb{P}}$ (see (2.11)) and $\tilde{\gamma}$ defined as in (2.10). Recall also that, under both $\mathbb{P}^*$ and $\widetilde{\mathbb{P}}$, W^* is still a standard Brownian motion. We will also make use of the decompositions (2.14)-(2.17). The proof of the following result is presented in Appendix B.

Theorem 4.2. *In the setting of Section 2, the at-the-money European call option price is such that:*

$$\lim_{t\to 0} t^{\frac{Y}{2}-1}\Big(t^{-\frac{1}{2}}\frac{1}{S_0}\,\mathbb{E}(S_t - S_0)_+ - \sigma\mathbb{E}^*(W_1^*)_+\Big) = \frac{C\sigma^{1-Y}}{Y(Y-1)}\mathbb{E}^*\left(|W_1^*|^{1-Y}\right). \tag{4.3}$$

Remark 4.3. As well known, the $(1-Y)$-centered moment of a standard normal distribution is given by (see, e.g., (25.6) in [37]):

$$\mathbb{E}^*\left(|W_1^*|^{1-Y}\right) = \frac{2^{1-Y}}{\sqrt{\pi}}\Gamma\left(1 - \frac{Y}{2}\right).$$

Thus, the second-order asymptotic behavior of the ATM call option price (1.1) in short-time takes the form

$$\frac{1}{S_0}\,\mathbb{E}(S_t - S_0)^+ = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + o\left(t^{\frac{3-Y}{2}}\right), \qquad (t \to 0), \tag{4.4}$$

with

$$d_1 = \frac{\sigma}{\sqrt{2\pi}}, \qquad d_2 = \frac{2^{1-Y}}{\sqrt{\pi}}\Gamma\left(1 - \frac{Y}{2}\right)\frac{C\sigma^{1-Y}}{Y(Y-1)}. \tag{4.5}$$

Intuitively, the first-order term only synthesizes the information about the continuous volatility parameter σ, while the second-order term incorporates also the information on the tail index parameter Y and the overall jump-intensity parameter C. However, these two-terms do not reflect the relative intensities of negative or positive jumps (controlled by the parameters G and M). This fact suggests that it could be necessary to develop a third-order approximation as described below.

Remark 4.4. The proof of Theorem 4.2 (See Appendix B) also provides higher order asymptotics of the ATM call option price under the generalized CGMY model. In fact, as mentioned in the proof, the second integral in the decomposition of B_t is asymptotically equivalent to $(\tilde{\gamma}/2)\sqrt{t}$, while the last integral is clearly asymptotically equivalent to $(-\sigma^2/4)\sqrt{t}$. Then, it remains to analyze the first integral A_t in (B.4), which in the proof of Theorem 4.2 is decomposed into $I_1(t)$, $I_2(t)$ and $I_3(t)$. For $I_1(t)$, it can be shown that the second order term of $J_{12}(t,y)$ is $O(t^{2-Y})$ while the first order of $J_{11}(t,y)$ is $O(\sqrt{t})$. For $I_2(t)$, the first term in the decomposition (B.18) is $o(\sqrt{t})$, while the second order term is $O(t^{2-Y})$. Finally, for $I_3(t)$, the second order of $J_{31}^{(2)}(t,y)$ and $J_{32}^{(2)}(t,y)$ is $O(t^{2-Y})$, while all the other terms in the decomposition of $I_3(t)$ are of order $o(\sqrt{t})$. Therefore, as $t \to 0$,

$$\frac{1}{S_0}\,\mathbb{E}(S_t - S_0)^+ = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + d_3 t + o(t), \qquad 1 < Y \leq \frac{3}{2}, \tag{4.6}$$

$$\frac{1}{S_0}\,\mathbb{E}(S_t - S_0)^+ = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + d_3 t^{\frac{5}{2}-Y} + o(t^{\frac{5}{2}-Y}), \qquad \frac{3}{2} < Y < 2, \tag{4.7}$$

where d_3 can be explicitly derived.

The next proposition gives the small-time asymptotic behavior for the ATM Black-Scholes implied volatility, again denoted by $\hat{\sigma}(t)$, under the generalized CGMY model. Unlike the pure-jump case, we can only derive the first order asymptotics using Theorem 4.2. In fact, the first order term of the ATM call option price under the generalized CGMY model is the same as the one under the Black-Scholes model. The second order term of $\hat{\sigma}_t$ requires higher order asymptotics of the ATM call option price. The proof is deferred to Appendix B.

Proposition 4.5. *Under the exponential CGMY model (1.2) with non-zero Brownian component, the implied volatility $\hat{\sigma}$ is such that:*

$$\hat{\sigma}(t) = \sigma + \frac{C 2^{\frac{3}{2}-Y} \sigma^{1-Y}}{Y(Y-1)} \Gamma\left(1 - \frac{Y}{2}\right) t^{1-\frac{Y}{2}} + o\left(t^{1-\frac{Y}{2}}\right), \quad t \to 0. \tag{4.8}$$

5 Numerical examples

In this part, we assess the performance of the previous approximations through a detailed numerical analysis.

5.1 The numerical methods

Let us first select a suitable numerical method to compute the ATM option prices by considering two methods: Inverse Fourier Transform (IFT) and Monte Carlo (MC).

Before introducing the IFT method, let us set some notations. The characteristic function corresponding to the Black-Scholes model with volatility Σ is given by

$$\varphi_t^{BS,\Sigma}(u) = \exp\left(-\frac{\Sigma^2 t}{2}\left(v^2 + iv\right)\right).$$

The corresponding call option price at the log-moneyness $k = \log(S_0/K)$ under the Black-Scholes model with volatility Σ is denoted by $C_{BS}^{\Sigma}(k)$; that is,

$$C_{BS}^{\Sigma}(k) = S_0 e^{-rt} \mathbb{E}(e^{(r-\Sigma^2/2)t + \Sigma W_t} - e^k)_+.$$

Let us also recall that the characteristic function under the generalized CGMY model is denoted by φ_t (see (2.2)) and let us denote the corresponding call option price at log-moneyness k by $C(k)$. The IFT method is based on the following inversion formula (see Section 11.1.3 in [8]):

$$z_T(k) := C(k) - C_{BS}^{\Sigma}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \zeta_T(v) dv, \tag{5.1}$$

where

$$\zeta_T(v) := e^{ivr} \frac{\varphi_T(v-i) - \varphi_T^{BS,\Sigma}(v-i)}{iv(1+iv)}. \tag{5.2}$$

In our case, we fix $r = 0$ and, since we are only interested in ATM option prices, we set $k = 0$. In order to compute numerically the integral in (5.1), we use the Simpson's rule:

$$z_T(0) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_T(v) dv = \Delta \sum_{m=0}^{P-1} w_m \zeta_T(v_m),$$

with $\Delta = Q/(P-1)$, $v_m = -Q/2 + m\Delta$, and $w_0 = 1/2$, $w_{2\ell-1} = 4/3$, and $w_{2\ell} = 2/3$, for $\ell = 1, \ldots, P/2$.

We also consider a Monte Carlo method based on the risk-neutral option price representation under the probability measure $\widetilde{\mathbb{P}}$. Under this probability measure and using the notation (2.15) as well as the relations (2.14) and (2.17), we have:

$$\begin{aligned}\mathbb{E}(e^{X_T}-1)_+ &= \mathbb{E}^*\left(e^{-X_T}\left(e^{X_T}-1\right)_+\right) = \widetilde{\mathbb{E}}\left(e^{-U_T}\left(1-e^{-X_T}\right)_+\right) \\ &= \widetilde{\mathbb{E}}\left(e^{-M^*\bar{U}_T^+ + G^*\bar{U}_T^- - \eta T}\left(1-e^{-\bar{U}_T^+ - \bar{U}_T^- - T\tilde{\gamma} - \sigma W_T^*}\right)_+\right),\end{aligned}$$

which can be easily computed by Monte Carlo method using the fact that, under $\widetilde{\mathbb{P}}$, the variables $\bar{U}_T^+$ and $-\bar{U}_T^-$ are independent Y-stable random variables with scale, skewness, and location parameters $TC|\cos(\pi Y/2)|\Gamma(-Y)$, 1, and 0, respectively. Standard simulation methods are available to generate stable random variables. We consider the following set of parameters for the CGMY component:

$$C = 0.5, \quad G = 2, \quad M = 3.6, \quad Y = 1.5.$$

Figure 1 compares the first- and second-order approximations as given in Remarks 3.3 and 4.3 to the prices based on the Inverse Fourier Transform (IFT-based price) and the Monte Carlo method (MC-based price) under both the pure-jump case and the generalized CGMY case with $\sigma = 0.4$. For the MC-based price, we use $100,000$ simulations, while for the IFT-based method, we use $P = 2^{14}$ and $Q = 800$. As it can be seen, it is not easy to integrate numerically the characteristic function (5.2) since in this case T is quite small and, therefore, the characteristic functions φ_T and $\varphi_T^{BS,\Sigma}$ are quite flat. The Monte Carlo method turns out to be much more accurate and faster.

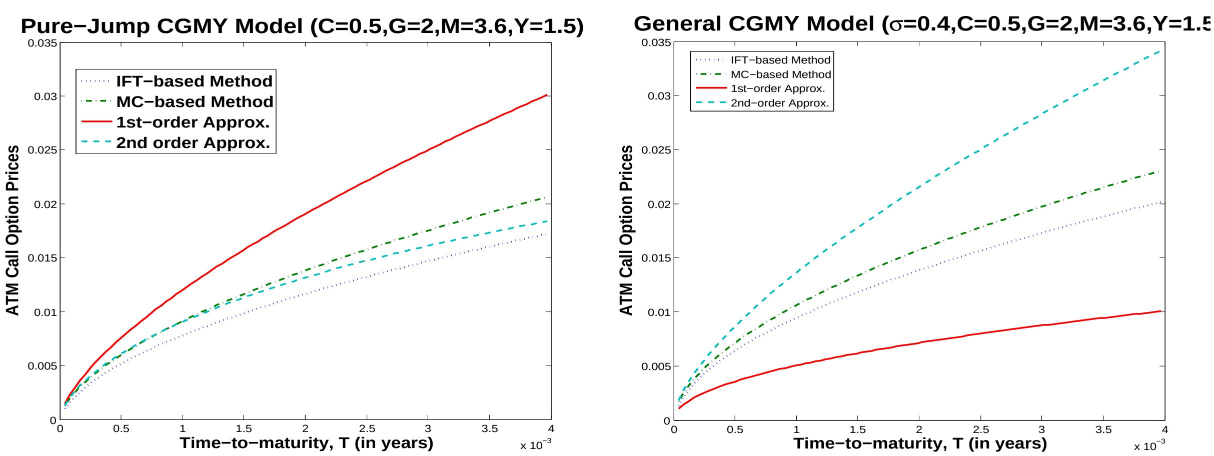

Figure 1: Comparisons of ATM call option prices for two methods (Inverse Fourier Transform and Monte-Carlo method) with the first- and second-order approximations. MC-based price is based on $100,000$ simulations while the IFT-based method is based on the parameter values $P = 2^{14}$ and $Q = 800$. The parameter σ in the generalized CGMY model is set to be 0.1.

5.2 Results for different parameter settings

Here, we investigate the performance of the approximations for different settings of parameters:

1. Figure 2 compares the 1st- and 2nd-order approximations with the MC prices for different values of C, fixing the values of all the other parameters. In the pure-jump case, the 2nd order approximation is significantly better for

moderately small values of C, but for larger values of C, this is not the case unless T is extremely small. For a nonzero continuous component, the 1st order approximation is extremely bad as it only takes into account the parameter σ.

2. Figure 3 compares the 1st- and 2nd-order approximations with the MC prices for different values of Y, fixing the values of all the other parameters. In both cases, the 2nd order approximation is significantly better for values of Y around 1.5, which is consistent with the observation that $|d_2| \to \infty$ as $Y \to 1$ or $Y \to 2$. For a nonzero continuous component, the 1st order approximation is again extremely bad as compared to the 2nd order approximation.

3. In the left panel of Figure 4, we analyze the effect of the relative intensities of negative jumps compared to positive jumps in the pure-jump CGMY case. That is, we fix the values M to be 4 and consider different values for G. As expected, since the first order approximation does not take into account this information, the 2nd-order approximation performs significantly better.

4. In the right panel of Figure 4, we analyze the effect of the volatility of the continuous component in the generalized CGMY case. The 2nd order approximation is, in general, much better than the 1st-order approximation and, interestingly enough, the quality of the 2nd order approximations improves as the values of σ increases. In fact, it seems that the 2nd-order approximation and the MC prices collapse to a steady curve as σ increases.

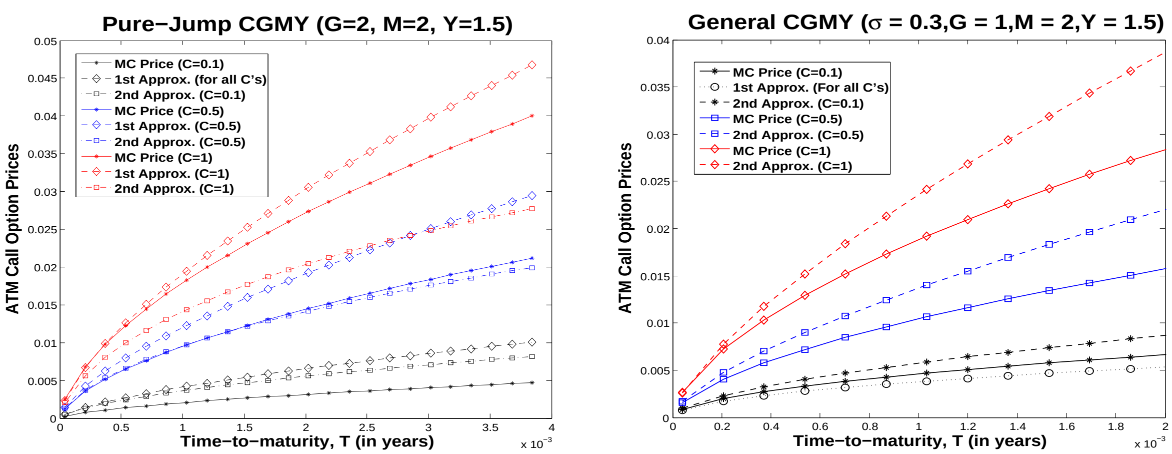

Figure 2: Comparisons of ATM call option prices with the short-time approximations for different values of the jump intensity parameter C.

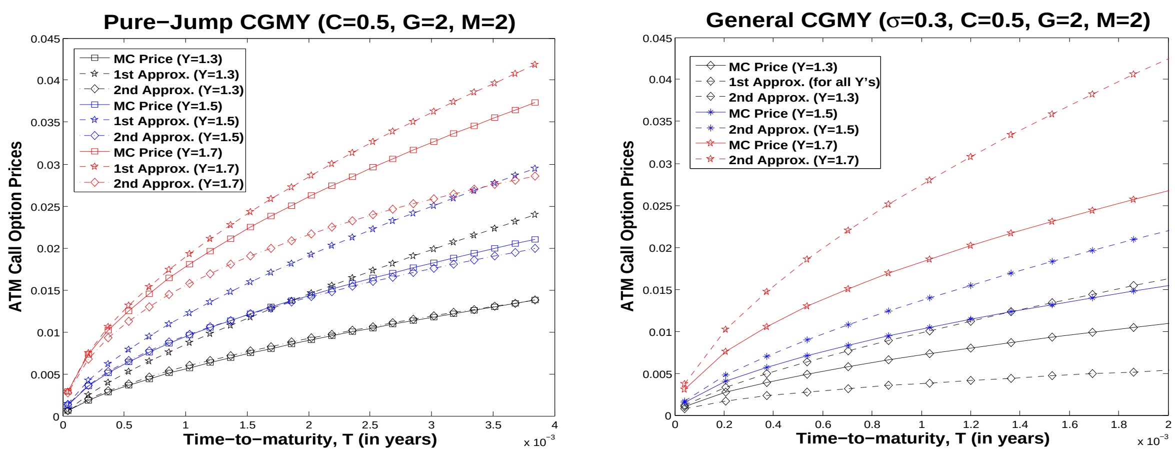

Figure 3: Comparisons of ATM call option prices with the short-time approximations for different values of the tail-heaviness parameter Y.

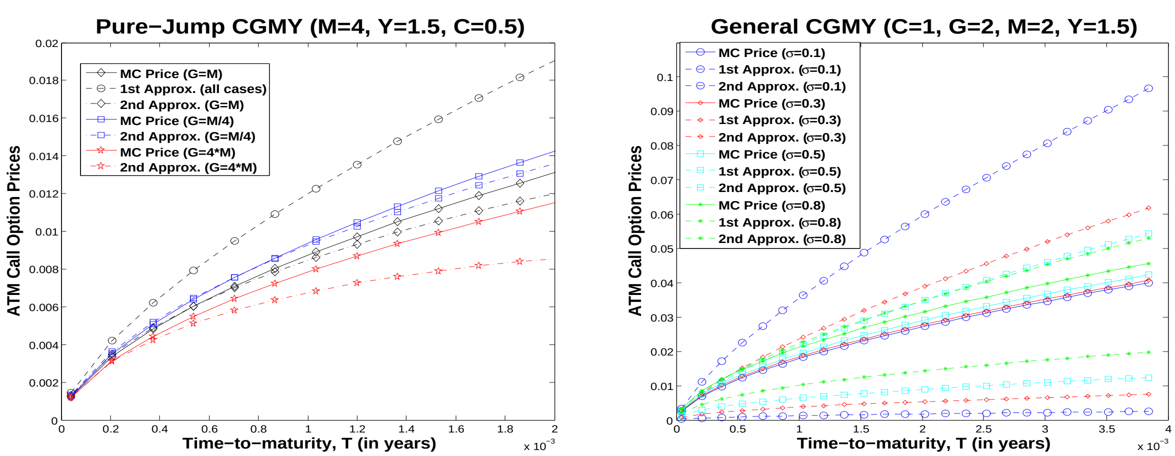

Figure 4: Comparisons of ATM call option prices with the short-time approximations for different values of G or M and different values of the volatility parameter σ.

A Proofs of Section 3: Pure-jump CGMY model

For simplicity, throughout this section, we fix $S_0 = 1$.

Proof of Lemma 3.1.

From (1.6), we have

$$t^{-1/Y}\mathbb{E}(S_t - S_0)^+ = t^{-1/Y}\mathbb{P}^*(X_t \geq E) = t^{-1/Y}\int_0^\infty e^{-x}\mathbb{P}^*(X_t \geq x)dx$$
$$= \int_0^\infty e^{-t^{1/Y}u}\mathbb{P}^*(t^{-1/Y}X_t \geq u)du. \tag{A.1}$$

Next, using the change of probability measure (2.13),

$$\mathbb{P}^*(t^{-1/Y}X_t \geq u) = \mathbb{E}^*\left(\mathbf{1}_{\{t^{-1/Y}X_t \geq u\}}\right) = \widetilde{\mathbb{E}}\left(e^{-U_t}\mathbf{1}_{\{t^{-1/Y}X_t \geq u\}}\right),$$

and, moreover, since $\sigma = 0$, (2.11), and (2.14),

$$\mathbb{P}^*(t^{-1/Y}X_t \geq u) = e^{-\eta t}\widetilde{\mathbb{E}}\left(e^{-\widetilde{U}_t}\mathbf{1}_{\{t^{-1/Y}Z_t \geq u - \tilde{\gamma}t^{1-1/Y}\}}\right),$$

with $\widetilde{U}_t := M^*\bar{U}_t^+ - G^*\bar{U}_t^-$. By the self-similarity property (2.12),

$$\mathbb{P}^*(t^{-1/Y}X_t \geq u) = e^{-\eta t}\widetilde{\mathbb{E}}\left(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1 \geq u - \tilde{\gamma}t^{1-1/Y}\}}\right). \tag{A.2}$$

Thus, plugging (A.2) into (A.1),

$$t^{-1/Y}\mathbb{E}(S_t - S_0)^+ = \int_0^\infty e^{-t^{1/Y}u - \eta t}\,\widetilde{\mathbb{E}}\left(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1 \geq u - \tilde{\gamma}t^{1-1/Y}\}}\right)du, \tag{A.3}$$

and, changing variables $\left(v = u - \tilde{\gamma}t^{1-1/Y}\right)$, the result follows. □

Proof of Theorem 3.2.

To begin with, we assume that $M - 1 = G$, so that $\tilde{\gamma} = 0$ (see (2.10)) and, in light of Lemma 3.1,

$$t^{-1/Y}\mathbb{E}(S_t - S_0)^+ = e^{-\eta t}\int_0^\infty e^{-t^{1/Y}v}\widetilde{\mathbb{E}}\left(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1 \geq v\}}\right)dv. \tag{A.4}$$

The general case is resolved in Lemma A.1 below. Let

$$D(t) := t^{-1/Y}\mathbb{E}(S_t - S_0)^+ - \widetilde{\mathbb{E}}(Z_1^+),$$

which can be written as

$$\begin{aligned}
D(t) :=& \int_0^\infty e^{-t^{1/Y}v}\widetilde{\mathbb{E}}\left(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1 \geq v\}}\right)dv - \widetilde{\mathbb{E}}(Z_1^+) \\
&+ (e^{-\eta t} - 1)\widetilde{\mathbb{E}}(Z_1^+) \\
&+ (e^{-\eta t} - 1)\left(\int_0^\infty e^{-t^{1/Y}v}\widetilde{\mathbb{E}}\left(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1 \geq v\}}\right)dv - \widetilde{\mathbb{E}}(Z_1^+)\right) \\
=:& D_1(t) + D_2(t) + D_3(t).
\end{aligned} \tag{A.5}$$

We will show that

$$t^{1/Y-1}D_1(t) \longrightarrow \vartheta + \eta, \qquad \text{as} \quad t \to 0, \tag{A.6}$$

while it is clear that $D_3(t) = o(D_1(t))$ and $t^{1/Y-1}D_2(t) = o(1)$, as $t \to 0$. First, note that in light of (2.26),

$$\begin{aligned} D_1(t) &= \widetilde{\mathbb{E}}\left(e^{-t^{1/Y}\widetilde{U}_1}\int_0^{Z_1^+} e^{-t^{1/Y}v}dv\right) - \widetilde{\mathbb{E}}\big(Z_1^+\big) \\ &= t^{-1/Y}\left(\widetilde{\mathbb{E}}\Big(e^{-t^{1/Y}\widetilde{U}_1}\Big) - \widetilde{\mathbb{E}}\Big(e^{-t^{1/Y}\left(\widetilde{U}_1+Z_1^+\right)}\Big)\right) - \widetilde{\mathbb{E}}(Z_1^+) \\ &= t^{-1/Y}\left(e^{\eta t} - \widetilde{\mathbb{E}}\Big(e^{-t^{1/Y}\left(\widetilde{U}_1+Z_1^+\right)}\Big)\right) - \widetilde{\mathbb{E}}(Z_1^+). \end{aligned} \tag{A.7}$$

Thus,

$$\begin{aligned} t^{1/Y-1}D_1(t) &= t^{1/Y-1}\left(\frac{e^{\eta t}-1}{t^{1/Y}} + \frac{1-\widetilde{\mathbb{E}}(e^{-t^{1/Y}(Z_1^+ + \widetilde{U}_1)})}{t^{1/Y}} - \widetilde{\mathbb{E}}(Z_1^+ + \widetilde{U}_1)\right) \\ &= \frac{e^{\eta t}-1}{t} + \widetilde{\mathbb{E}}\int_0^{Z_1^+ + \widetilde{U}_1} \frac{e^{-t^{1/Y}v}-1}{t^{1-1/Y}}dv\mathbf{1}_{\{Z_1^+ + \widetilde{U}_1 \geq 0\}} - \widetilde{\mathbb{E}}\int_{Z_1^+ + \widetilde{U}_1}^0 \frac{e^{-t^{1/Y}v}-1}{t^{1-1/Y}}dv\mathbf{1}_{\{Z_1^+ + \widetilde{U}_1 \leq 0\}} \\ &= \underbrace{\frac{e^{\eta t}-1}{t}}_{D_{11}(t)} + \underbrace{\int_0^\infty \frac{e^{-t^{1/Y}v}-1}{t^{1-1/Y}}\widetilde{\mathbb{P}}(Z_1^+ + \widetilde{U}_1 \geq v)dv}_{D_{12}(t)} - \underbrace{\int_0^\infty \frac{e^{t^{1/Y}v}-1}{t^{1-1/Y}}\widetilde{\mathbb{P}}(Z_1^+ + \widetilde{U}_1 \leq -v)dv}_{D_{13}(t)}. \end{aligned} \tag{A.8}$$

Clearly,

$$D_{11}(t) \to \eta, \quad \text{as} \quad t \to 0. \tag{A.9}$$

Next, for $D_{13}(t)$, note that

$$\widetilde{\mathbb{P}}(Z_1^+ + \widetilde{U}_1 \leq -v) \leq \widetilde{\mathbb{P}}(\widetilde{U}_1 \leq -v) \leq \frac{\widetilde{\mathbb{E}}\big(e^{-\widetilde{U}_1}\big)}{e^v} = e^{\eta - v}, \tag{A.10}$$

and, since $0 < e^y - 1 \leq ye^y$ for $y > 0$,

$$\frac{|e^{t^{1/Y}v}-1|}{t^{1-1/Y}}\,\widetilde{\mathbb{P}}(Z_1^+ + \widetilde{U}_1 \leq -v) \leq t^{2/Y-1}e^\eta v e^{(t^{1/Y}-1)v} \leq e^\eta v e^{-v/2},$$

for $t > 0$ small enough. The dominated convergence theorem then implies that

$$D_{13}(t) \to 0, \quad \text{as} \quad t \to 0. \tag{A.11}$$

Finally, let us analyze the term $D_{12}(t)$. Changing variables $\big(u = t^{1/Y}v\big)$,

$$D_{12}(t) = t^{-1}\int_0^\infty (e^{-u}-1)\widetilde{\mathbb{P}}\Big(Z_1^+ + \widetilde{U}_1 \geq t^{-1/Y}u\Big)du,$$

and, from (2.22) and (2.23), there exists, as shown in Appendix C, a constant $\tilde{\kappa} < \infty$ such that

$$\frac{1}{t}\widetilde{\mathbb{P}}\Big(Z_1^+ + \widetilde{U}_1 \geq t^{-1/Y}u\Big) \leq \tilde{\kappa}u^{-Y}, \tag{A.12}$$

for any $0 < t \leq 1$ and $u > 0$. Hence, by the dominated convergence theorem,

$$\lim_{t\to 0} D_{12}(t) = \int_0^\infty (e^{-u}-1)\lim_{t\to 0}\Big(t^{-1}\widetilde{\mathbb{P}}\Big(Z_1^+ + \widetilde{U}_1 \geq t^{-1/Y}u\Big)\Big)\,du. \tag{A.13}$$

To find $\lim_{t\to 0} t^{-1}\widetilde{\mathbb{P}}(Z_1^+ + \widetilde{U}_1 \geq t^{-1/Y}u)$, note that

$$\widetilde{\mathbb{P}}\left(Z_1^+ + \widetilde{U}_1 \geq \frac{u}{t^{1/Y}}\right) = \underbrace{\widetilde{\mathbb{P}}\Big(Z_1 + \widetilde{U}_1 \geq \frac{u}{t^{1/Y}}, Z_1 \geq 0\Big)}_{I_1} + \underbrace{\widetilde{\mathbb{P}}\Big(\widetilde{U}_1 \geq \frac{u}{t^{1/Y}}, Z_1 < 0\Big)}_{I_2}.$$

Then,

$$
\begin{aligned}
I_1 &= \widetilde{\mathbb{P}}\Big((M^*+1)\bar{U}_1^+ - (G^*-1)\bar{U}_1^- \geq t^{-1/Y}u, \bar{U}_1^+ + \bar{U}_1^- \geq 0\Big) \\
&= \widetilde{\mathbb{P}}\Big(\bar{U}_1^+ \geq \frac{t^{-1/Y}u + (G^*-1)\bar{U}_1^-}{M^*+1} \geq -\bar{U}_1^-\Big) + \widetilde{\mathbb{P}}\Big(\bar{U}_1^+ \geq -\bar{U}_1^- \geq \frac{t^{-1/Y}u + (G^*-1)\bar{U}_1^-}{M^*+1}\Big) \\
&= \widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq \frac{t^{-1/Y}u + (G^*-1)\bar{U}_1^-}{M^*+1}, -\bar{U}_1^- \leq \frac{t^{-1/Y}u}{M^*+G^*}\right) + \widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq -\bar{U}_1^- \geq \frac{t^{-1/Y}u}{M^*+G^*}\right) \\
&:= I_{1,1}(t) + I_{1,2}(t).
\end{aligned}
$$

By the independence of $\bar{U}_1^+$ and $\bar{U}_1^-$ and the estimate (2.20),

$$
I_{1,2}(t) \leq \widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq \frac{t^{-1/Y}u}{M^*+G^*}\right)\widetilde{\mathbb{P}}\left(-\bar{U}_1^- \geq \frac{t^{-1/Y}u}{M^*+G^*}\right) = O(t^2), \tag{A.14}
$$

as $t \to 0$. Note also that

$$
\frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq \frac{t^{-\frac{1}{Y}}u - (G^*-1)y}{M^*+1}\right)\mathbf{1}_{\{y \leq \frac{t^{-1/Y}u}{M^*+G^*}\}} \leq \frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq \frac{t^{-\frac{1}{Y}}u - (G^*-1)\frac{t^{-1/Y}u}{M^*+G^*}}{M^*+1}\right)\mathbf{1}_{\{y \leq \frac{t^{-1/Y}u}{M^*+G^*}\}} \leq \frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq \frac{t^{-\frac{1}{Y}}u}{M^*+G^*}\right),
$$

and so, recalling that $p(1,y)$ denotes the density of $-\bar{U}_1^-$, then using (2.20), Lemma 2.1, and the dominated convergence theorem,

$$
\begin{aligned}
\lim_{t\to 0}\frac{1}{t}I_{1,1}(t) &= \lim_{t\to 0}\frac{1}{t}\int_{\mathbb{R}} p(1,y)\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq \frac{t^{-1/Y}u - (G^*-1)y}{M^*+1}\right)\mathbf{1}_{\{y \leq \frac{t^{-1/Y}u}{M^*+G^*}\}}dy \\
&= \int_{\mathbb{R}} p(1,y)\lim_{t\to 0}\frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq \frac{t^{-1/Y}u - (G^*-1)y}{M^*+1}\right)\mathbf{1}_{\{y \leq \frac{t^{-1/Y}u}{M^*+G^*}\}}dy \\
&= \int_{\mathbb{R}} p(1,y)\lim_{t\to 0}\frac{1}{t}\,\frac{tC(M^*+1)^Y}{Y(u-t^{1/Y}(G^*-1)y)^Y}\mathbf{1}_{\{y \leq \frac{t^{-1/Y}u}{M^*+G^*}\}}dy \\
&= \frac{C(M^*+1)^Y}{Yu^Y}.
\end{aligned} \tag{A.15}
$$

Similarly,

$$
\begin{aligned}
I_2 &= \widetilde{\mathbb{P}}\Big(M^*\bar{U}_1^+ - G^*\bar{U}_1^- \geq t^{-1/Y}u, \bar{U}_1^+ + \bar{U}_1^- < 0\Big) \\
&= \widetilde{\mathbb{P}}\left(-\bar{U}_1^- \geq \frac{t^{-1/Y}u - M^*U_1^+}{G^*} > \bar{U}_1^+\right) + \widetilde{\mathbb{P}}\left(-\bar{U}_1^- > \bar{U}_1^+ \geq \frac{t^{-1/Y}u - M^*\bar{U}_1^+}{G^*}\right) \\
&= \widetilde{\mathbb{P}}\left(-\bar{U}_1^- \geq \frac{t^{-1/Y}u - M^*\bar{U}_1^+}{G^*}, \bar{U}_1^+ < \frac{t^{-1/Y}u}{M^*+G^*}\right) + \widetilde{\mathbb{P}}\left(-\bar{U}_1^- > \bar{U}_1^+ \geq \frac{t^{-1/Y}u}{M^*+G^*}\right) \\
&:= I_{2,1}(t) + I_{2,2}(t).
\end{aligned}
$$

Again, as in (A.14),

$$
\widetilde{\mathbb{P}}\left(-\bar{U}_1^- > \bar{U}_1^+ \geq \frac{t^{-1/Y}u}{M^*+G^*}\right) = O(t^2), \qquad (t \to 0), \tag{A.16}
$$

and since

$$
\frac{1}{t}\widetilde{\mathbb{P}}\Big(-\bar{U}_1^- \geq \frac{t^{-\frac{1}{Y}}u - M^*y}{G^*}\Big)\mathbf{1}_{\{y \leq \frac{t^{-1/Y}u}{M^*+G^*}\}} \leq \frac{1}{t}\widetilde{\mathbb{P}}\Big(-\bar{U}_1^- \geq \frac{t^{-\frac{1}{Y}}u - M^*\frac{t^{-1/Y}u}{M^*+G^*}}{G^*}\Big)\mathbf{1}_{\{y \leq \frac{t^{-1/Y}u}{M^*+G^*}\}} \leq \frac{1}{t}\widetilde{\mathbb{P}}\Big(-\bar{U}_1^- \geq \frac{t^{-\frac{1}{Y}}u}{M^*+G^*}\Big),
$$

by (2.20), Lemma 2.1, and the dominated convergence theorem,

$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t}I_{2,1}(t) &= \lim_{t\to 0} t^{-1}\int_{\mathbb{R}} p(1,y)\widetilde{\mathbb{P}}\left(-\bar{U}_1^- \geq \frac{t^{-1/Y}u - M^*y}{G^*}\right)\mathbf{1}_{\{y\leq \frac{t^{-1/Y}u}{M^*+G^*}\}}dy \\
&= \int_{\mathbb{R}} p(1,y)\lim_{t\to 0} t^{-1}\widetilde{\mathbb{P}}\left(-\bar{U}_1^- \geq \frac{t^{-1/Y}u - M^*y}{G^*}\right)\mathbf{1}_{\{y\leq \frac{t^{-1/Y}u}{M^*+G^*}\}}dy \\
&= \int_{\mathbb{R}} p(1,y)\lim_{t\to 0}\frac{1}{t}\,\frac{tC(G^*)^Y}{Y(u-t^{1/Y}M^*y)^Y}\mathbf{1}_{\{y\leq \frac{t^{-1/Y}u}{M^*+G^*}\}}dy \\
&= \frac{C(G^*)^Y}{Yu^Y}.
\end{aligned} \tag{A.17}$$

Combining (A.8), (A.9), (A.11), and (A.14)-(A.17) implies that

$$\lim_{t\to 0} t^{1/Y-1}D_1(t) = -\frac{C}{Y}\Big((M^*+1)^Y + (G^*)^Y\Big)\int_0^\infty \left(1-e^{-v}\right)v^{-Y}dv + \eta.$$

Finally, we use the following identity (see p. 84 in [37]):

$$\int_0^\infty (e^{-v}-1)v^{-Y}dv = \Gamma(1-Y) = -Y\Gamma(-Y).$$

This concludes the proof. □

Lemma A.1. *If $\tilde{\gamma}\neq 0$ in (3.1), then*

$$\lim_{t\to 0} t^{\frac{1}{Y}-1}\left(t^{-\frac{1}{Y}}\frac{1}{S_0}\mathbb{E}(S_t-S_0)^+ - \widetilde{\mathbb{E}}(Z_1^+)\right) = \vartheta + \eta + \frac{\tilde{\gamma}}{2}. \tag{A.18}$$

Proof: Without loss of generality, fix $S_0 = 1$ and also assume that $\tilde{\gamma} > 0$, the case $\tilde{\gamma} < 0$ being similar. Then, using (3.1),

$$\begin{aligned}
&\lim_{t\to 0} t^{1/Y-1}\left(t^{-1/Y}\mathbb{E}(S_t-S_0)^+ - \widetilde{\mathbb{E}}(Z_1^+)\right) \\
&= \lim_{t\to 0} t^{1/Y-1}\left(e^{-(\tilde{\gamma}+\eta)t}\int_0^\infty e^{-t^{1/Y}v}\widetilde{\mathbb{E}}\left(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1\geq v\}}\right)dv - \widetilde{\mathbb{E}}(Z_1^+)\right) \\
&\quad + \lim_{t\to 0} t^{1/Y-1}e^{-(\tilde{\gamma}+\eta)t}\int_{-\tilde{\gamma}t^{1-1/Y}}^0 e^{-t^{1/Y}v}\widetilde{\mathbb{E}}\left(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1\geq v\}}\right)dv \\
&:= \widetilde{D}_{11}(t) + \widetilde{D}_{12}(t).
\end{aligned}$$

As in the proof of (A.6), it can be shown that

$$\lim_{t\to 0}\widetilde{D}_{11}(t) = \vartheta + \eta. \tag{A.19}$$

For $\widetilde{D}_{12}(t)$, changing variables $\left(u = t^{1/Y-1}v\right)$, we have

$$\widetilde{D}_{12}(t) = e^{-(\tilde{\gamma}+\eta)t}\int_{-\tilde{\gamma}}^0 e^{-tu}\widetilde{\mathbb{E}}\Big(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1\geq t^{1-1/Y}u\}}\Big)du = e^{-(\tilde{\gamma}+\eta)t}\int_{-\tilde{\gamma}}^0 g_t^{(1)}(u)du + e^{-(\tilde{\gamma}+\eta)t}\int_{-\tilde{\gamma}}^0 g_t^{(2)}(u)du,$$

where

$$g_t^{(1)}(u) := e^{-tu}\widetilde{\mathbb{E}}\Big(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1\geq t^{1-1/Y}u\}}\mathbf{1}_{\{\widetilde{U}_1\geq 0\}}\Big), \quad g_t^{(2)}(u) := e^{-tu}\widetilde{\mathbb{E}}\Big(e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1\geq t^{1-1/Y}u\}}\mathbf{1}_{\{\widetilde{U}_1< 0\}}\Big).$$

Since $Y\in(1,2)$, it is easy to see that, for $-\tilde{\gamma}\leq u\leq 0$ and $0\leq t\leq 1$,

$$\left|\,g_t^{(1)}(u)\right| \leq \widetilde{\mathbb{P}}\big(Z_1\geq -\tilde{\gamma},\, \widetilde{U}_1\geq 0\big), \qquad \left|\,g_t^{(2)}(u)\right| \leq e^{\tilde{\gamma}}\widetilde{\mathbb{E}}\Big(e^{-\widetilde{U}_1}\mathbf{1}_{\{Z_1\geq -\tilde{\gamma}\}}\Big),$$

and

$$e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1\geq t^{1-1/Y}u\}}\mathbf{1}_{\{\widetilde{U}_1\geq 0\}}\leq \mathbf{1}_{\{Z_1\geq u\}},\quad e^{-t^{1/Y}\widetilde{U}_1}\mathbf{1}_{\{Z_1\geq t^{1-1/Y}u\}}\mathbf{1}_{\{\widetilde{U}_1<0\}}\leq e^{-\widetilde{U}_1}\mathbf{1}_{\{Z_1\geq u\}}.$$

By the dominated convergence theorem, it follows that

$$\lim_{t\to 0}\widetilde{D}_{12}(t)=\int_{-\tilde{\gamma}}^{0}\Big(\widetilde{\mathbb{P}}\big(Z_1\geq 0,\,\widetilde{U}_1\geq 0\big)+\widetilde{\mathbb{P}}\big(Z_1\geq 0,\,\widetilde{U}_1<0\big)\Big)\,du=\frac{\tilde{\gamma}}{2}. \tag{A.20}$$

Combining the previous limit with (A.19) leads to (A.18). □

Proof of Proposition 3.5.

The small-time asymptotic behavior of the ATM call-option price $C_{BS}(t,\sigma)$ at maturity t under the Black-Scholes model with volatility σ and zero interest rates is known to be such that (fixing for simplicity $S_0=1$)

$$C_{BS}(t,\sigma)=\frac{\sigma}{\sqrt{2\pi}}t^{1/2}-\frac{\sigma^3}{24\sqrt{2\pi}}t^{3/2}+O(t^{5/2}),\qquad t\to 0; \tag{A.21}$$

see, e.g., [18, Corollary 3.4]). To derive the small-time asymptotics for the implied volatility, we need an analogous result to (A.21) when σ is replaced by $\hat{\sigma}(t)$. To show such a formula, the following representation due [33, Lemma 3.1] will be useful

$$C_{BS}(t,\sigma)=F(\sigma\sqrt{t})\quad\text{with}\quad F(\theta):=\int_0^\theta \Phi'\left(\frac{v}{2}\right)dv=\frac{1}{\sqrt{2\pi}}\int_0^\theta \exp\left(-v^2/8\right)dv,$$

together with the following Taylor expansion for F at $\theta=0$ (see [33, Lemma 5.1])

$$F(\theta)=\frac{1}{\sqrt{2\pi}}\theta-\frac{1}{24\sqrt{2\pi}}\theta^3+O(\theta^5),\quad \theta\to 0.$$

Then, since $\hat{\sigma}(t)\to 0$ as $t\to 0$ (see, e.g., [38, Proposition 5]), we conclude that

$$C_{BS}(t,\hat{\sigma}(t))=\frac{\hat{\sigma}(t)}{\sqrt{2\pi}}t^{1/2}-\frac{\hat{\sigma}(t)^3}{24\sqrt{2\pi}}t^{3/2}+O\left(\left(\hat{\sigma}(t)t^{1/2}\right)^5\right),\quad\text{as}\quad t\to 0. \tag{A.22}$$

Returning to the proof of Proposition 3.5, by equating (3.4) and (A.22) and comparing the first order terms,

$$\widetilde{\mathbb{E}}(Z_1^+)t^{1/Y}\sim\frac{\hat{\sigma}(t)}{\sqrt{2\pi}}t^{1/2},\qquad t\to 0,$$

and, therefore,

$$\hat{\sigma}(t)\sim\sqrt{2\pi}\widetilde{\mathbb{E}}(Z_1^+)t^{\frac{1}{Y}-\frac{1}{2}}:=\sigma_1 t^{\frac{1}{Y}-\frac{1}{2}},\qquad t\to 0. \tag{A.23}$$

Next, set $\tilde{\sigma}(t)=\hat{\sigma}(t)-\sigma_1 t^{\frac{1}{Y}-\frac{1}{2}}$. By comparing the first and second order terms in (3.4) with the first term in (A.22) (noting that the second order term in (A.22) is $o(t)$),

$$\frac{C\Gamma(-Y)}{2}\left((M-1)^Y-M^Y-(G+1)^Y+G^Y\right)t\sim\frac{\tilde{\sigma}(t)}{\sqrt{2\pi}}t^{1/2},\qquad t\to 0.$$

Hence, $\tilde{\sigma}(t)\to 0$ as $t\to 0$, and moreover

$$\tilde{\sigma}(t)\sim\sqrt{\frac{\pi}{2}}C\Gamma(-Y)\left((M-1)^Y-M^Y-(G+1)^Y+G^Y\right)t^{1/2},\qquad t\to 0. \tag{A.24}$$

Combining (A.23) and (A.24) finishes the proof. □

B Proofs of Section 4: CGMY model with Brownian component

Proof of Proposition 4.1.

From (1.6), note that

$$t^{-1/2}\mathbb{E}(S_t - S_0)^+ = t^{-1/2}S_0\mathbb{P}^*(X_t \geq E) = t^{-1/2}S_0\int_0^\infty e^{-x}\mathbb{P}^*(X_t \geq x)dx = S_0\int_0^\infty e^{-\sqrt{t}u}\mathbb{P}^*(t^{-1/2}X_t \geq u)du.$$

Now for any $u \geq 0$ and $0 < t \leq 1$,

$$\begin{aligned} e^{-\sqrt{t}u}\mathbb{P}^*(t^{-1/2}X_t \geq u) &\leq \mathbb{P}^*(t^{-1/2}\sigma W_t^* \geq u/2) + \mathbb{P}^*(t^{-1/2}L_t^* \geq u/2) \\ &= \mathbb{P}^*(\sigma W_1^* \geq u/2) + \mathbb{P}^*(t^{-1/2}L_t^* \geq u/2). \end{aligned} \tag{B.1}$$

Clearly the first term in (B.1) is integrable on $[0, \infty)$. To estimate the second term, applying the change of probability measure (2.13) and using the self-similarity property (2.12) of Z, we obtain

$$\mathbb{P}^*(t^{-1/2}L_t^* \geq u/2) = \widetilde{\mathbb{E}}\left(e^{-U_t}\mathbf{1}_{\{t^{-1/2}Z_t \geq u/2 - \tilde{\gamma}\sqrt{t}\}}\right) = \widetilde{\mathbb{E}}\left(e^{-U_t}\mathbf{1}_{\{Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}}u/2 - \tilde{\gamma}t^{1-\frac{1}{Y}}\}}\right).$$

Pick $1 < q < Y$ and $q' > 1$ such that $q^{-1} + {q'}^{-1} = 1$, then by Hölder's inequality and (2.26),

$$\begin{aligned} \widetilde{\mathbb{E}}\left(e^{-U_t}\mathbf{1}_{\{Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}}u/2 - \tilde{\gamma}t^{1-\frac{1}{Y}}\}}\right) &\leq \left(\widetilde{\mathbb{E}}\Big(e^{-q'U_t}\Big)\right)^{1/q'}\left(\widetilde{\mathbb{P}}\Big(Z_1 \geq \frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\Big)\right)^{1/q} \\ &= e^{-\eta t}\left(\widetilde{\mathbb{E}}\Big(e^{-q'\tilde{U}_t}\Big)\right)^{1/q'}\left(\widetilde{\mathbb{P}}\Big(Z_1 \geq \frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\Big)\right)^{1/q} \\ &= e^{-\eta t}e^{\eta {q'}^{Y-1}t}\left(\widetilde{\mathbb{P}}\Big(Z_1 \geq \frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\Big)\right)^{1/q} \\ &\leq \exp\left\{\eta\Big({q'}^{Y-1} - 1\Big)\right\}\left(\widetilde{\mathbb{P}}\Big(Z_1 \geq \frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\Big)\right)^{1/q}, \end{aligned}$$

since as given in (2.16), $\eta > 0$. If $\tilde{\gamma} \leq 0$, then by (2.23), for any $u > 0$ and $0 < t \leq 1$,

$$\left(\widetilde{\mathbb{P}}\Big(Z_1 \geq \frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\Big)\right)^{1/q} \leq \left(\frac{8\kappa}{\big(\frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\big)^Y}\right)^{1/q} \leq \frac{(16\kappa)^{1/q}}{u^{Y/q}},$$

and thus,

$$\left(\widetilde{\mathbb{P}}\Big(Z_1 \geq \frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\Big)\right)^{1/q} \leq \min\left(1, \frac{(16\kappa)^{1/q}}{u^{Y/q}}\right),$$

which is integrable on $[0, +\infty)$. On the other hand, if $\tilde{\gamma} > 0$,

$$\begin{aligned} \widetilde{\mathbb{P}}\Big(Z_1 \geq \frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\Big) &= \widetilde{\mathbb{P}}\Big(Z_1 \geq \frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\Big)\mathbf{1}_{\{u > 2\tilde{\gamma}\sqrt{t}\}} + \widetilde{\mathbb{P}}\Big(Z_1 \geq \frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\Big)\mathbf{1}_{\{u \leq 2\tilde{\gamma}\sqrt{t}\}} \\ &\leq \frac{16\kappa}{u^Y} + \frac{2^Y\tilde{\gamma}^Y}{u^Y}, \end{aligned}$$

and thus

$$\left(\widetilde{\mathbb{P}}\Big(Z_1 \geq \frac{t^{\frac{1}{2}-\frac{1}{Y}}u}{2} - \tilde{\gamma}t^{1-\frac{1}{Y}}\Big)\right)^{1/q} \leq \min\left(1, \frac{(16\kappa + 2^Y\tilde{\gamma}^Y)^{1/q}}{u^{Y/q}}\right),$$

which is also integrable on $[0,+\infty)$. Therefore, by the dominated convergence theorem,

$$\lim_{t\to 0} t^{-1/2}\mathbb{E}(S_t - S_0)_+ = S_0 \int_0^\infty \mathbb{P}^*(\sigma W_1^* \geq u)du = S_0\sigma\mathbb{E}^*(W_1^*)_+.$$

The proof is now complete. $\square$

Proof of Theorem 4.2.

For simplicity, we fix $S_0 = 1$. Recalling that $X_t = L_t^* + \sigma W_t^*$ under $\mathbb{P}^*$ and using (1.6), the self-similarity of W^*, and the change of variable $u = t^{-1/2}x$, it follows that

$$B_t := t^{-1/2}\mathbb{E}(S_t - S_0)_+ - \sigma\mathbb{E}^*(W_1^*)_+ = \int_0^\infty e^{-\sqrt{t}u}\mathbb{P}^*\left(\sigma W_1^* \geq u - t^{-\frac{1}{2}}L_t\right)du - \int_0^\infty \mathbb{P}^*(\sigma W_1^* \geq u)du.$$

Next, by changing the probability measure to $\widetilde{\mathbb{P}}$ and using that $L_t^* = Z_t + \tilde{\gamma}t$, $U_t = \widetilde{U}_t + \eta t$, and the change of variable $y = u - t^{1/2}\tilde{\gamma}$ in the first integral above, we get

$$\begin{aligned}
B_t &= \int_{-t^{1/2}\tilde{\gamma}}^\infty e^{-\sqrt{t}y-\tilde{\gamma}t}\,\widetilde{\mathbb{E}}\, e^{-\widetilde{U}_t-\eta t}\mathbf{1}_{\{\sigma W_1^*\geq y-t^{-\frac{1}{2}}Z_t\}}dy - \int_0^\infty \widetilde{\mathbb{E}}e^{-\widetilde{U}_t-\eta t}\mathbf{1}_{\{\sigma W_1^*\geq u\}}du\\
&= e^{-(\eta+\tilde{\gamma})t}\int_0^\infty e^{-\sqrt{t}y}\left(\widetilde{\mathbb{E}}\, e^{-\widetilde{U}_t}\mathbf{1}_{\{\sigma W_1^*\geq y-t^{-\frac{1}{2}}Z_t\}} - \widetilde{\mathbb{E}}e^{-\widetilde{U}_t}\mathbf{1}_{\{\sigma W_1^*\geq y\}}\right)dy \qquad \text{(B.2)}\\
&\quad + e^{-(\eta+\tilde{\gamma})t}\int_{-t^{1/2}\tilde{\gamma}}^0 e^{-\sqrt{t}y}\widetilde{\mathbb{E}}\, e^{-\widetilde{U}_t}\mathbf{1}_{\{\sigma W_1^*\geq y-t^{-\frac{1}{2}}Z_t\}}dy\\
&\quad + \int_0^\infty \left(e^{-\tilde{\gamma}t-\sqrt{t}y}-1\right)\mathbb{P}^*(\sigma W_1^*\geq y)dy.
\end{aligned}$$

The last term above is clearly $O(\sqrt{t})$ as $t\to 0$, while the second term above can be shown to be asymptotically equivalent to $(\tilde{\gamma}/2)\sqrt{t}$ by arguments analogous to those of (A.20). Thus, we only need to analyze the term in (B.2) that we denote A_t and that can be written as follows in light of the self-similarity property of Z and $\widetilde{U}$:

$$A_t = e^{-\tilde{\eta}t}\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\int_0^\infty e^{-\sqrt{t}y}\left(\mathbf{1}_{\{\sigma W_1^*\geq y-t^{\frac{1}{Y}-\frac{1}{2}}Z_1\}} - \mathbf{1}_{\{\sigma W_1^*\geq y\}}\right)dy\right), \qquad \text{(B.3)}$$

where we had denoted $\tilde{\eta} := \eta + \tilde{\gamma}$. To analyze the asymptotic behavior of A_t, we decompose it into the following three terms:

$$\begin{aligned}
A_t &= e^{-\tilde{\eta}t}\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\{W_1^*\geq 0,\sigma W_1^*+t^{\frac{1}{Y}-\frac{1}{2}}Z_1\geq 0\}}\int_{\sigma W_1^*}^{\sigma W_1^*+t^{\frac{1}{Y}-\frac{1}{2}}Z_1} e^{-\sqrt{t}y}dy\right)\\
&\quad - e^{-\tilde{\eta}t}\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\{0\leq\sigma W_1^*\leq -t^{\frac{1}{Y}-\frac{1}{2}}Z_1\}}\int_0^{\sigma W_1^*} e^{-\sqrt{t}y}dy\right)\\
&\quad + e^{-\tilde{\eta}t}\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\{0\leq -\sigma W_1^*\leq t^{\frac{1}{Y}-\frac{1}{2}}Z_1\}}\int_0^{\sigma W_1^*+t^{\frac{1}{Y}-\frac{1}{2}}Z_1} e^{-\sqrt{t}y}dy\right)\\
&:= I_1(t) - I_2(t) + I_3(t). \qquad \text{(B.4)}
\end{aligned}$$

We analyze each of three terms above in the following three steps:

Step 1. We first analyze the behavior of $I_1(t)$. Since $(Z_t)_{t\geq 0}$ and $(W^*_t)_{t\geq 0}$ are independent,

$$\begin{aligned}
I_1(t) &= e^{-\tilde{\eta}t}\widetilde{\mathbb{E}}\left(\mathbf{1}_{\{W_1^*\geq 0,\sigma W_1^*+t^{\frac{1}{Y}-\frac{1}{2}}Z_1\geq 0\}}\frac{e^{-t^{\frac{1}{Y}}\widetilde{U}_1}-e^{-t^{\frac{1}{Y}}(\widetilde{U}_1+Z_1)}}{\sqrt{t}}e^{-\sqrt{t}\sigma W_1^*}\right)\\
&= e^{-\tilde{\eta}t}\int_0^\infty \widetilde{\mathbb{E}}\Big(\mathbf{1}_{\{Z_1\geq -t^{\frac{1}{2}-\frac{1}{Y}}y\}}\frac{e^{-t^{\frac{1}{Y}}\widetilde{U}_1}(1-e^{-t^{\frac{1}{Y}}Z_1})}{\sqrt{t}}\Big)e^{-\sqrt{t}y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\\
&:= e^{-\tilde{\eta}t}\int_0^\infty J_1(t,y)e^{-\sqrt{t}y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy.
\end{aligned}\tag{B.5}$$

Using that the distribution of Z_1 is symmetric under $\widetilde{\mathbb{P}}$ (hence, $\widetilde{\mathbb{E}}Z_1=0$), $J_1(t,y)$ is then decomposed as:

$$\begin{aligned}
J_1(t,y) &= \widetilde{\mathbb{E}}\left(\mathbf{1}_{\{Z_1\geq -t^{\frac{1}{2}-\frac{1}{Y}}y\}}\left(\frac{e^{-t^{\frac{1}{Y}}\widetilde{U}_1}-e^{-t^{\frac{1}{Y}}(\widetilde{U}_1+Z_1)}}{\sqrt{t}}-t^{\frac{1}{Y}-\frac{1}{2}}Z_1\right)\right)+t^{\frac{1}{Y}-\frac{1}{2}}\widetilde{\mathbb{E}}\Big(Z_1\mathbf{1}_{\{Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}y\}}\Big)\\
&:= J_{11}(t,y)+J_{12}(t,y).
\end{aligned}\tag{B.6}$$

Let us first consider $J_{12}(t,y)$. By (2.24), it follows that, for any $0<t\leq 1$ and $y>0$,

$$t^{\frac{Y}{2}-1}J_{12}(t,y)\leq \lambda y^{1-Y},\tag{B.7}$$

for some $\lambda<\infty$ (see Appendix C for the verification of this claim). Moreover, for any fixed $y>0$,

$$t^{\frac{Y}{2}-1}J_{12}(t,y)=t^{\frac{Y}{2}+\frac{1}{Y}-\frac{3}{2}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}y}^\infty u p_Z(u)du = t^{\frac{Y}{2}-\frac{1}{Y}-\frac{1}{2}}\int_y^\infty w p_Z(t^{\frac{1}{2}-\frac{1}{Y}}w)dw.$$

Using (2.25),

$$t^{\frac{Y}{2}-\frac{1}{Y}-\frac{1}{2}}wp_Z(t^{\frac{1}{2}-\frac{1}{Y}}w)\leq (C+1)w^{-Y},$$

for t small enough and all $w\geq y$. Therefore, by the dominated convergence theorem and, in light of (2.25), we get:

$$\begin{aligned}
&\lim_{t\to 0}t^{\frac{Y}{2}-1}e^{-\tilde{\eta}t}\int_0^\infty J_{12}(t,y)e^{-\sqrt{t}y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\\
&= \int_0^\infty\left(\lim_{t\to 0}t^{\frac{Y}{2}-1}J_{12}(t,y)\right)\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\\
&= \int_0^\infty\left(\int_y^\infty w\left(\lim_{t\to 0}t^{\frac{Y}{2}-\frac{1}{Y}-\frac{1}{2}}p_Z(t^{\frac{1}{2}-\frac{1}{Y}}w)\right)dw\right)\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\\
&= \int_0^\infty\left(\int_y^\infty Cw^{-Y}dw\right)\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy=\frac{C}{Y-1}\int_0^\infty y^{1-Y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy.
\end{aligned}\tag{B.8}$$

For $J_{11}(t,y)$, note that

$$\begin{aligned}
J_{11}(t,y) &= t^{\frac{1}{Y}-\frac{1}{2}}\widetilde{\mathbb{E}}\left(\mathbf{1}_{\{Z_1\geq 0\}}\int_{\widetilde{U}_1}^{\widetilde{U}_1+Z_1}\Big(e^{-t^{\frac{1}{Y}}u}-1\Big)du\right)-t^{\frac{1}{Y}-\frac{1}{2}}\widetilde{\mathbb{E}}\left(\mathbf{1}_{\{-t^{\frac{1}{2}-\frac{1}{Y}}y\leq Z_1\leq 0\}}\int_{\widetilde{U}_1+Z_1}^{\widetilde{U}_1}\Big(e^{-t^{\frac{1}{Y}}u}-1\Big)du\right)\\
&= t^{\frac{1}{Y}-\frac{1}{2}}\int_{\mathbb{R}}\Big(e^{-t^{\frac{1}{Y}}u}-1\Big)\widetilde{\mathbb{P}}\big(Z_1\geq 0,\widetilde{U}_1\leq u\leq \widetilde{U}_1+Z_1\big)du\\
&\quad -t^{\frac{1}{Y}-\frac{1}{2}}\int_{\mathbb{R}}\Big(e^{-t^{\frac{1}{Y}}u}-1\Big)\widetilde{\mathbb{P}}\big(-t^{\frac{1}{2}-\frac{1}{Y}}y\leq Z_1\leq 0,\widetilde{U}_1+Z_1\leq u\leq \widetilde{U}_1\big)du\\
&= t^{\frac{1}{Y}-\frac{1}{2}}\int_0^\infty\Big(e^{-t^{\frac{1}{Y}}u}-1\Big)\Big(\widetilde{\mathbb{P}}\big(Z_1\geq 0,\widetilde{U}_1\leq u\leq \widetilde{U}_1+Z_1\big)-\widetilde{\mathbb{P}}\big(-t^{\frac{1}{2}-\frac{1}{Y}}y\leq Z_1\leq 0,\widetilde{U}_1+Z_1\leq u\leq \widetilde{U}_1\big)\Big)du\\
&\quad +t^{\frac{1}{Y}-\frac{1}{2}}\int_{-\infty}^0\Big(e^{-t^{\frac{1}{Y}}u}-1\Big)\Big(\widetilde{\mathbb{P}}\big(Z_1\geq 0,\widetilde{U}_1\leq u\leq \widetilde{U}_1+Z_1\big)-\widetilde{\mathbb{P}}\big(-t^{\frac{1}{2}-\frac{1}{Y}}y\leq Z_1\leq 0,\widetilde{U}_1+Z_1\leq u\leq \widetilde{U}_1\big)\Big)dx.
\end{aligned}$$

Next, change variable back to $x = t^{1/Y}u$:

$$
\begin{aligned}
J_{11}(t,y) = \frac{1}{\sqrt{t}}\int_0^{\infty}\left(e^{-x}-1\right)\Bigg(&\widetilde{\mathbb{P}}\big(Z_1 \geq 0, \widetilde{U}_1 \leq t^{-\frac{1}{Y}}x \leq \widetilde{U}_1 + Z_1\big)\\
&- \widetilde{\mathbb{P}}\big(-t^{\frac{1}{2}-\frac{1}{Y}}y \leq Z_1 \leq 0, \widetilde{U}_1 + Z_1 \leq t^{-\frac{1}{Y}}x \leq \widetilde{U}_1\big)\Bigg)du\\
&+ \frac{1}{\sqrt{t}}\int_{-\infty}^{0}\left(e^{-x}-1\right)\Bigg(\widetilde{\mathbb{P}}\big(Z_1 \geq 0, \widetilde{U}_1 \leq t^{-\frac{1}{Y}}x \leq \widetilde{U}_1 + Z_1\big)\\
&- \widetilde{\mathbb{P}}\big(-t^{\frac{1}{2}-\frac{1}{Y}}y \leq Z_1 \leq 0, \widetilde{U}_1 + Z_1 \leq t^{-\frac{1}{Y}}x \leq \widetilde{U}_1\big)\Bigg)dx.
\end{aligned} \tag{B.9}
$$

For $t > 0$ and $y > 0$, set

$$
T_1(t,x,y) := \widetilde{\mathbb{P}}\big(Z_1 \geq 0, \widetilde{U}_1 \leq t^{-\frac{1}{Y}}x \leq \widetilde{U}_1 + Z_1\big), \quad T_2(t,x,y) := \widetilde{\mathbb{P}}\big(-t^{\frac{1}{2}-\frac{1}{Y}}y \leq Z_1 \leq 0, \widetilde{U}_1 + Z_1 \leq t^{-\frac{1}{Y}}x \leq \widetilde{U}_1\big).
$$

By (A.12), there exists $\tilde{\kappa} > 0$ such that for any $x > 0$ and $0 < t \leq 1$,

$$
T_1(t,x,y) \leq \widetilde{\mathbb{P}}\big(t^{-\frac{1}{Y}}x \leq \widetilde{U}_1 + Z_1\big) \leq \tilde{\kappa}tx^{-Y}. \tag{B.10}
$$

Hence,

$$
\begin{aligned}
0 &\leq e^{-\eta t}t^{\frac{Y}{2}-1}\int_0^{\infty}\int_0^{\infty}\frac{(1-e^{-x})}{\sqrt{t}}T_1(t,x,y)dx\frac{e^{-\sqrt{t}y}e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\\
&\leq t^{\frac{Y-3}{2}}\int_0^{\infty}\int_0^{\infty}(1-e^{-x})T_1(t,x,y)dx\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\\
&\leq \kappa t^{\frac{Y-1}{2}}\int_0^{\infty}\int_0^{\infty}(1-e^{-x})x^{-Y}dx\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy \longrightarrow 0,
\end{aligned} \tag{B.11}
$$

as $t \to 0$, since $Y > 1$. Similarly, using $\widetilde{U}_1 = M^*\bar{U}_1^+ - G^*\bar{U}_1^-$ and Lemma 2.1, for any $0 < t \leq 1$ and $x, y > 0$, we have

$$
\begin{aligned}
T_2(t,x,y) &\leq \widetilde{\mathbb{P}}\big(t^{-\frac{1}{Y}}x \leq \widetilde{U}_1\big)\\
&\leq \widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq \frac{t^{-\frac{1}{Y}}x}{2M^*}\right) + \widetilde{\mathbb{P}}\left(-\bar{U}_1^- \geq \frac{t^{-\frac{1}{Y}}x}{2G^*}\right)\\
&\leq 2\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \geq \frac{t^{-\frac{1}{Y}}x}{2(M^*+G^*)}\right)\\
&\leq 8\kappa t(M^*+G^*)^Y x^{-Y}.
\end{aligned} \tag{B.12}
$$

Therefore,

$$
\lim_{t\to 0}\frac{e^{-\eta t}}{t^{1-\frac{Y}{2}}}\int_0^{\infty}\int_0^{\infty}\frac{(1-e^{-x})}{\sqrt{t}}T_2(t,x,y)dx\frac{e^{-\sqrt{t}y}e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy = 0. \tag{B.13}
$$

For $x < 0$, since $\bar{U}_1^+$ and $-\bar{U}_1^-$ are identically distributed, we proceed in the proof as follows:

$$
\begin{aligned}
T_1(t,x,y) &\leq \widetilde{\mathbb{P}}\big(\widetilde{U}_1 \leq t^{-\frac{1}{Y}}x\big) = \widetilde{\mathbb{P}}\big(M^*\bar{U}_1^+ - G^*\bar{U}_1^- \leq t^{-\frac{1}{Y}}x\big)\\
&\leq \widetilde{\mathbb{P}}\big(\bar{U}_1^+ \leq (2M^*)^{-1}t^{-\frac{1}{Y}}x\big) + \widetilde{\mathbb{P}}\big(-\bar{U}_1^- \leq (2G^*)^{-1}t^{-\frac{1}{Y}}x\big)\\
&\leq 2\widetilde{\mathbb{P}}\big(\bar{U}_1^+ \leq (2M^*+2G^*)^{-1}t^{-\frac{1}{Y}}x\big)\\
&\leq 2\widetilde{\mathbb{E}}(e^{-\bar{U}_1^+})\exp\left(\frac{t^{-\frac{1}{Y}}x}{2(M^*+G^*)}\right),
\end{aligned} \tag{B.14}
$$

which is again independent of y and integrable on $[0,\infty)$ when multiplied by $e^{-x}-1$. Moreover,

$$
\begin{aligned}
0 &\le t^{\frac{Y-3}{2}} \int_{-\infty}^{0} (e^{-x}-1) T_1(t,x,y)dx \\
&\le 2\widetilde{\mathbb{E}}(e^{-\bar{U}_1^+}) t^{\frac{Y-3}{2}} \int_{-\infty}^{0} \left(\exp\left(\frac{t^{-\frac{1}{Y}}x}{2(M^*+G^*)} - x \right) - \exp\left(\frac{t^{-\frac{1}{Y}}x}{2(M^*+G^*)} \right) \right) dx \\
&= 2\widetilde{\mathbb{E}}(e^{-\bar{U}_1^+}) t^{\frac{Y-3}{2}} \frac{t^{\frac{2}{Y}}(M^*+G^*)^2}{1-2t^{\frac{1}{Y}}(M^*+G^*)}.
\end{aligned}
$$

Hence, by the dominated convergence theorem,

$$
\begin{aligned}
0 &\le e^{-\eta t} t^{\frac{Y}{2}-1} \int_0^\infty \int_{-\infty}^0 \frac{e^{-x}-1}{\sqrt{t}} T_1(t,x,y)dx \frac{e^{-\sqrt{t}y} e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\
&\le 2\widetilde{\mathbb{E}}(e^{-\bar{U}_1^+}) t^{\frac{Y-3}{2}} \frac{t^{\frac{2}{Y}}(M^*+G^*)^2}{1-2t^{\frac{1}{Y}}(M^*+G^*)} \int_0^\infty \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \longrightarrow 0,
\end{aligned}
\tag{B.15}
$$

as $t \to 0$ since $2/Y > 1 > (3-Y)/2$, for $1 < Y < 2$. Similarly, since $\widetilde{U}_1 + Z_1 = M\bar{U}_1^+ - G\bar{U}_1^-$, it follows from (B.14) that

$$
T_2(t,x,y) \le \widetilde{\mathbb{P}}\big(M\bar{U}_1^+ - G\bar{U}_1^- \le t^{-\frac{1}{Y}}x\big) \le 2\widetilde{\mathbb{E}}(e^{-\bar{U}_1^+}) \exp\left(\frac{t^{-\frac{1}{Y}}x}{2(M+G)} \right).
$$

Therefore,

$$
\lim_{t\to 0} e^{-\eta t} t^{\frac{Y}{2}-1} \int_0^\infty \int_{-\infty}^0 \frac{(1-e^{-x})}{\sqrt{t}} T_2(t,x,y)dx \frac{e^{-\sqrt{t}y} e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy = 0. \tag{B.16}
$$

Combining (B.8), (B.11), (B.13), (B.15) and (B.16), we obtain

$$
\lim_{t\to 0} t^{\frac{Y}{2}-1} I_1(t) = \frac{C}{Y-1} \int_0^\infty y^{1-Y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \tag{B.17}
$$

Step 2. Next, we analyze the asymptotic behavior of $I_2(t)$. Using the independence of $(Z_t)_{t\ge 0}$ and $(W_t^*)_{t\ge 0}$,

$$
\begin{aligned}
I_2(t) &= e^{-\tilde{\eta}t} \widetilde{\mathbb{E}} \left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1} \mathbf{1}_{\{0 \le \sigma W_1^* \le -t^{\frac{1}{Y}-\frac{1}{2}} Z_1\}} \frac{1-e^{-\sqrt{t}\sigma W_1^*}}{\sqrt{t}} \right) \\
&= e^{-\tilde{\eta}t} \int_0^\infty \widetilde{\mathbb{E}}\Big(e^{-t^{\frac{1}{Y}}\widetilde{U}_1} \mathbf{1}_{\{Z_1 \le -t^{\frac{1}{2}-\frac{1}{Y}} y\}} \Big) \frac{1-e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\
&= e^{-\tilde{\eta}t} \int_0^\infty \widetilde{\mathbb{E}}\Big((e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - 1) \mathbf{1}_{\{Z_1 \le -t^{\frac{1}{2}-\frac{1}{Y}} y\}} \Big) \frac{1-e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\
&\quad + e^{-\tilde{\eta}t} \int_0^\infty \widetilde{\mathbb{P}}\Big(Z_1 \le -t^{\frac{1}{2}-\frac{1}{Y}} y \Big) \frac{1-e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy.
\end{aligned}
\tag{B.18}
$$

By (2.23) and the symmetry of Z_1,

$$
\widetilde{\mathbb{P}}\Big(Z_1 \le -t^{\frac{1}{2}-\frac{1}{Y}} y \Big) \frac{1-e^{-\sqrt{t}y}}{\sqrt{t}} = \widetilde{\mathbb{P}}\Big(Z_1 \ge t^{\frac{1}{2}-\frac{1}{Y}} y \Big) \frac{1-e^{-\sqrt{t}y}}{\sqrt{t}} \le 8\kappa t^{1-\frac{Y}{2}} y^{1-Y} \le 8\kappa y^{1-Y},
$$

which, when multiplied by $\exp(-y^2/(2\sigma^2))$, becomes integrable on $[0,\infty)$. Hence, by (2.24) and the dominated convergence theorem,

$$
\begin{aligned}
&\lim_{t\to 0} t^{\frac{Y}{2}-1} e^{-\tilde{\eta}t} \int_0^\infty \widetilde{\mathbb{P}}\Big(Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}} y\Big) \frac{1-e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\
&= \int_0^\infty \lim_{t\to 0} t^{\frac{Y}{2}-1} \widetilde{\mathbb{P}}\Big(Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}} y\Big) y \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\
&= \int_0^\infty \lim_{t\to 0} t^{\frac{Y}{2}-1} \widetilde{\mathbb{P}}\Big(Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}} y\Big) y \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\
&= \int_0^\infty \lim_{t\to 0} t^{\frac{Y}{2}-1} \frac{C}{Y} (t^{\frac{1}{2}-\frac{1}{Y}} y)^{-Y} y \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\
&= \frac{C}{Y} \int_0^\infty y^{1-Y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy.
\end{aligned}
\tag{B.19}
$$

To find the asymptotic behavior of the first integral in (B.18), we decompose it as:

$$
\begin{aligned}
\widetilde{\mathbb{E}}\Big((e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - 1)\mathbf{1}_{\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}} y\}}\Big) &= \widetilde{\mathbb{E}}\Big(\mathbf{1}_{\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}} y, \widetilde{U}_1 < 0\}} \int_{t^{\frac{1}{Y}}\widetilde{U}_1}^0 e^{-u} du\Big) \\
&\quad - \widetilde{\mathbb{E}}\Big(\mathbf{1}_{\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}} y, \widetilde{U}_1 \geq 0\}} \int_0^{t^{\frac{1}{Y}}\widetilde{U}_1} e^{-u} du\Big) \\
&:= J_{21}(t,y) + J_{22}(t,y).
\end{aligned}
\tag{B.20}
$$

For $J_{21}(t,y)$, note that for any $0 < t \leq 2^{-Y}(M^* + G^*)^{-Y}$ and $y \geq 0$,

$$
\begin{aligned}
J_{21}(t,y) &= \int_{-\infty}^0 e^{-x} \widetilde{\mathbb{P}}\Big(\bar{U}_1^+ + \bar{U}_1^- \leq -t^{\frac{1}{2}-\frac{1}{Y}} y, M^*\bar{U}_1^+ - G^*\bar{U}_1^- \leq t^{-\frac{1}{Y}} x\Big) dx \\
&\leq \int_{-\infty}^0 e^{-x} \widetilde{\mathbb{P}}\Big(M^*\bar{U}_1^+ - G^*\bar{U}_1^- \leq t^{-\frac{1}{Y}} x\Big) dx \\
&\leq 2\int_{-\infty}^0 e^{-x} \widetilde{\mathbb{P}}\Big(\bar{U}_1^+ \leq \frac{t^{-\frac{1}{Y}} x}{M^* + G^*}\Big) dx \\
&\leq 2\widetilde{\mathbb{E}}\big(e^{-\bar{U}_1^+}\big) \int_{-\infty}^0 e^{-x} \exp\left(\frac{t^{-\frac{1}{Y}} x}{M^* + G^*}\right) dx \\
&= 2\widetilde{\mathbb{E}}\big(e^{-\bar{U}_1^+}\big) \frac{t^{\frac{1}{Y}}(M^* + G^*)}{1 - t^{\frac{1}{Y}}(M^* + G^*)},
\end{aligned}
$$

which is independent of y. Since $1 - Y/2 < 1/2 < 1/Y$, for $1 < Y < 2$, by the dominated convergence theorem,

$$
\begin{aligned}
0 &\leq t^{\frac{Y}{2}-1} e^{-\eta t} \int_0^\infty J_{21}(t,y) \frac{1-e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\
&\leq t^{\frac{Y}{2}-1} 2\widetilde{\mathbb{E}}\big(e^{-\bar{U}_1^+}\big) \frac{t^{\frac{1}{Y}}(M^* + G^*)}{1 - t^{\frac{1}{Y}}(M^* + G^*)} \int_0^\infty y \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \longrightarrow 0.
\end{aligned}
\tag{B.21}
$$

We further decompose the second term $J_{22}(t,y)$ in (B.20) as:

$$
\begin{aligned}
J_{22}(t,y) &= \widetilde{\mathbb{E}}\Big((e^{-t^{\frac{1}{Y}}\widetilde{U}_1} - 1 + t^{\frac{1}{Y}}\widetilde{U}_1)\mathbf{1}_{\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}} y, \widetilde{U}_1 \geq 0\}}\Big) \\
&\quad - t^{\frac{1}{Y}} \widetilde{\mathbb{E}}\Big(\widetilde{U}_1 \mathbf{1}_{\{Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}} y, \widetilde{U}_1 \geq 0\}}\Big) \\
&:= J_{22}^{(1)}(t,y) - J_{22}^{(2)}(t,y).
\end{aligned}
\tag{B.22}
$$

Since $1-Y/2<1/2<1/Y$, for $1<Y<2$, it is easy to see that

$$J_{22}^{(2)}(t,y)\le t^{\frac{1}{Y}}\widetilde{\mathbb{E}}\big|\widetilde{U}_1\big|=O(t^{\frac{1}{Y}})=o(t^{1-\frac{Y}{2}}).\tag{B.23}$$

Moreover,

$$\begin{aligned}J_{22}^{(1)}(t,y)&=\widetilde{\mathbb{E}}\left(\int_0^{t^{\frac{1}{Y}}\widetilde{U}_1}(1-e^{-w})dw\cdot\mathbf{1}_{\{Z_1\le -t^{\frac{1}{2}-\frac{1}{Y}}y,\widetilde{U}_1\ge 0\}}\right)\\&=\int_0^\infty(1-e^{-w})\widetilde{\mathbb{P}}\Big(\widetilde{U}_1\ge t^{-\frac{1}{Y}}w,Z_1\le -t^{\frac{1}{2}-\frac{1}{Y}}y\Big)dw\\&\le\int_0^\infty(1-e^{-w})\widetilde{\mathbb{P}}\Big(\widetilde{U}_1\ge t^{-\frac{1}{Y}}w\Big)dw.\end{aligned}$$

Hence by (B.12) and the dominated convergence theorem,

$$0\le t^{\frac{Y}{2}-1}e^{-\tilde{\eta}t}\int_0^\infty J_{22}^{(1)}(t,y)\frac{1-e^{-\sqrt{t}y}}{\sqrt{t}}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\le 8\kappa(M^*{+}G^*)^Y t^{\frac{Y}{2}}\int_0^\infty\frac{1-e^{-w}}{w^{-Y}}dw\int_0^\infty y\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\to 0,\tag{B.24}$$

as $t\to 0$. Combining (B.19), (B.21), (B.23) and (B.24), we obtain

$$\lim_{t\to 0}t^{\frac{Y}{2}-1}I_2(t)=\frac{C}{Y}\int_0^\infty y^{1-Y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy.\tag{B.25}$$

Step 3. We finally analyze the behavior of $I_3(t)$. Note that

$$\begin{aligned}I_3(t)&=e^{-\tilde{\eta}t}\int_0^\infty\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\{Z_1\ge t^{\frac{1}{2}-\frac{1}{Y}}y\}}\frac{1-e^{\sqrt{t}y}e^{-t^{\frac{1}{Y}}Z_1}}{\sqrt{t}}\right)\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\\&=e^{-\tilde{\eta}t}\int_0^\infty\widetilde{\mathbb{E}}\Big(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\{Z_1\ge t^{\frac{1}{2}-\frac{1}{Y}}y\}}\Big)\frac{1-e^{\sqrt{t}y}}{\sqrt{t}}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\\&\quad+e^{-\tilde{\eta}t}\int_0^\infty\widetilde{\mathbb{E}}\left(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\mathbf{1}_{\{Z_1\ge t^{\frac{1}{2}-\frac{1}{Y}}y\}}\frac{1-e^{-t^{\frac{1}{Y}}Z_1}}{\sqrt{t}}\right)e^{\sqrt{t}y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy\\&:=e^{-\tilde{\eta}t}\int_0^\infty J_{31}(t,y)\frac{1-e^{\sqrt{t}y}}{\sqrt{t}}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy+e^{-\tilde{\eta}t}\int_0^\infty J_{32}(t,y)\frac{e^{\sqrt{t}y}e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy.\end{aligned}\tag{B.26}$$

We first investigate the asymptotic of $J_{31}(t,y)$ by decomposing it as:

$$\begin{aligned}J_{31}(t,y)&=\widetilde{\mathbb{E}}\Big(\big(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}-1\big)\mathbf{1}_{\{Z_1\ge t^{\frac{1}{2}-\frac{1}{Y}}y\}}\Big)+\widetilde{\mathbb{P}}\Big(Z_1\ge t^{\frac{1}{2}-\frac{1}{Y}}y\Big)\\&:=J_{31}^{(1)}(t,y)+J_{31}^{(2)}(t,y).\end{aligned}\tag{B.27}$$

By (2.23), it is easy to see that $J_{31}^{(2)}(t,y)\le 8\kappa t^{1-\frac{Y}{2}}y^{-Y}$, for any $0<t\le 1$ and $y\ge 0$. Hence, by (2.24) and the dominated convergence theorem,

$$\lim_{t\to 0}\frac{e^{-\tilde{\eta}t}}{t^{1-\frac{Y}{2}}}\int_0^\infty J_{31}^{(2)}(t,y)\frac{1-e^{\sqrt{t}y}}{\sqrt{t}}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy=-\frac{C}{Y}\int_0^\infty y^{1-Y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy.\tag{B.28}$$

Also, $J_{31}^{(1)}(t,y)$ can be further decomposed as:

$$\widetilde{\mathbb{E}}\Big(\big(e^{-t^{\frac{1}{Y}}\widetilde{U}_1}-1\big)\mathbf{1}_{\{Z_1\ge t^{\frac{1}{2}-\frac{1}{Y}}y,\widetilde{U}_1\ge 0\}}\Big)+\int_0^\infty e^u\widetilde{\mathbb{P}}\Big(Z_1\ge t^{\frac{1}{2}-\frac{1}{Y}}y,\widetilde{U}_1\le -t^{-\frac{1}{Y}}u\Big)du.$$

For any $u>0$, $y>0$ and $t>0$, let $T_3(t,u,y):=\widetilde{\mathbb{P}}\Big(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}y,\widetilde{U}_1\leq -t^{-\frac{1}{Y}}u\Big)$. It is easily seen that

$$\begin{aligned}
T_3(t,u,y)&\leq \widetilde{\mathbb{P}}\Big(\bar{U}_1^+\leq \frac{-t^{-\frac{1}{Y}}u}{2M^*}\Big)+\widetilde{\mathbb{P}}\Big(-\bar{U}_1^-\leq \frac{-t^{-\frac{1}{Y}}u}{2G^*}\Big)\\
&\leq 2\widetilde{\mathbb{P}}\Big(-\bar{U}_1^-\leq \frac{-t^{-\frac{1}{Y}}u}{2(M^*+G^*)}\Big)\\
&\leq 2\widetilde{\mathbb{E}}e^{\bar{U}_1^-}\exp\left(\frac{-t^{-\frac{1}{Y}}u}{2(M^*+G^*)}\right).
\end{aligned} \tag{B.29}$$

Moreover,

$$0\leq \widetilde{\mathbb{E}}\Big(\big(1-e^{-t^{\frac{1}{Y}}\widetilde{U}_1}\big)\mathbf{1}_{\{Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}y,\widetilde{U}_1\geq 0\}}\Big)\leq t^{\frac{1}{Y}}\widetilde{\mathbb{E}}\big|\widetilde{U}_1\big|. \tag{B.30}$$

Hence, for any $y\geq 0$ and $0<t\leq 2^{-Y}(M^*+G^*)^{-Y}$, and since $1-Y/2<1/2<1/Y$, for $1<Y<2$,

$$\begin{aligned}
t^{\frac{Y}{2}-1}|J_{31}^{(1)}(t,y)|&\leq t^{\frac{Y}{2}-1}2\widetilde{\mathbb{E}}e^{\bar{U}_1^-}\int_0^\infty \exp\left(\frac{-t^{-\frac{1}{Y}}u}{2(M^*+G^*)}+u\right)du+t^{\frac{1}{Y}+\frac{Y}{2}-1}\widetilde{\mathbb{E}}\big|\widetilde{U}_1\big|\\
&=t^{\frac{Y}{2}-1}\frac{4\widetilde{\mathbb{E}}\Big(e^{\bar{U}_1^-}\Big)t^{\frac{1}{Y}}(M^*+G^*)}{1-2t^{\frac{1}{Y}}(M^*+G^*)}+t^{\frac{1}{Y}+\frac{Y}{2}-1}\widetilde{\mathbb{E}}\big|\widetilde{U}_1\big|\longrightarrow 0,\qquad \text{as } t\to 0.
\end{aligned} \tag{B.31}$$

Since both control functions in (B.29) and (B.30) are independent of y, combining (B.28) and (B.31), and by the dominated convergence theorem,

$$\lim_{t\to 0}\frac{e^{-\tilde{\eta}t}}{t^{1-\frac{Y}{2}}}\int_0^\infty J_{31}(t,y)\frac{1-e^{\sqrt{t}y}}{\sqrt{t}}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy=-\frac{C}{Y}\int_0^\infty y^{1-Y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy. \tag{B.32}$$

Next, we decompose the quantity $J_{32}(t,y)$ defined in (B.26) as:

$$\begin{aligned}
J_{32}(t,y)&=\widetilde{\mathbb{E}}\left(\mathbf{1}_{\{Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}y\}}\Big(\frac{e^{-t^{\frac{1}{Y}}\widetilde{U}_1}-e^{-t^{\frac{1}{Y}}(Z_1+\widetilde{U}_1)}}{\sqrt{t}}-t^{\frac{1}{Y}-\frac{1}{2}}Z_1\Big)\right)\\
&\quad+\widetilde{\mathbb{E}}\Big(t^{\frac{1}{Y}-\frac{1}{2}}Z_1\mathbf{1}_{\{Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}y\}}\Big)\\
&:=J_{32}^{(1)}(t,y)+J_{32}^{(2)}(t,y).
\end{aligned} \tag{B.33}$$

Note that $J_{32}^{(2)}(t,y)$ is the same as $J_{12}(t,y)$ in (B.6), and thus the corresponding integral has an asymptotic behavior similar to (B.8):

$$\lim_{t\to 0}t^{\frac{Y}{2}-1}e^{-\tilde{\eta}t}\int_0^\infty \widetilde{\mathbb{E}}\Big(t^{\frac{1}{Y}-\frac{1}{2}}Z_1\mathbf{1}_{\{Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}y\}}\Big)e^{\sqrt{t}y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy=\frac{C}{Y-1}\int_0^\infty y^{1-Y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy. \tag{B.34}$$

Next, we further decompose $J_{32}^{(1)}(t,y)$ as:

$$\begin{aligned}
J_{32}^{(1)}(t,y)&=t^{\frac{1}{Y}-\frac{1}{2}}\widetilde{\mathbb{E}}\left(\mathbf{1}_{\{Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}y\}}\int_{\widetilde{U}_1}^{Z_1+\widetilde{U}_1}\big(e^{-t^{\frac{1}{Y}}u}-1\big)du\right)\\
&=t^{\frac{1}{Y}-\frac{1}{2}}\int_{\mathbb{R}}(e^{-t^{\frac{1}{Y}}u}-1)\widetilde{\mathbb{P}}\Big(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}y,\widetilde{U}_1\leq u\leq Z_1+\widetilde{U}_1\Big)du\\
&=\frac{1}{\sqrt{t}}\int_{-\infty}^0(e^{-x}-1)\widetilde{\mathbb{P}}\Big(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}y,\widetilde{U}_1\leq t^{-\frac{1}{Y}}x\leq Z_1+\widetilde{U}_1\Big)dx\\
&\quad+\frac{1}{\sqrt{t}}\int_0^\infty(e^{-x}-1)\widetilde{\mathbb{P}}\Big(Z_1\geq t^{\frac{1}{2}-\frac{1}{Y}}y,\widetilde{U}_1\leq t^{-\frac{1}{Y}}x\leq Z_1+\widetilde{U}_1\Big)dx.
\end{aligned}$$

Note that for $x>0$,

$$\widetilde{\mathbb{P}}\Big(Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}}y, \widetilde{U}_1 \leq t^{-\frac{1}{Y}}x \leq Z_1+\widetilde{U}_1\Big) \leq \widetilde{\mathbb{P}}\Big(t^{-\frac{1}{Y}}x \leq Z_1+\widetilde{U}_1\Big),$$

while for $x<0$,

$$\widetilde{\mathbb{P}}\Big(Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}}y, \widetilde{U}_1 \leq t^{-\frac{1}{Y}}x \leq Z_1+\widetilde{U}_1\Big) \leq \widetilde{\mathbb{P}}\Big(\widetilde{U}_1 \leq t^{-\frac{1}{Y}}x\Big).$$

Using the estimates (B.10) and (B.14), a proof as in getting (B.11) and (B.15) gives

$$\lim_{t\to 0}\frac{e^{-\tilde{\eta}t}}{t^{1-\frac{Y}{2}}}\int_0^\infty J_{32}^{(1)}(t,y)e^{\sqrt{t}y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy = 0. \tag{B.35}$$

Combining (B.32), (B.34) and (B.35), we have

$$\lim_{t\to 0} t^{\frac{Y}{2}-1}I_3(t) = \frac{C}{Y(Y-1)}\int_0^\infty y^{1-Y}\frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}dy. \tag{B.36}$$

Finally, from (B.4), (B.17), (B.25) and (B.36), and since $1-Y/2<1/2$, for $1<y<2$, we obtain (4.3), therefore finishing the proof. □

Proof of Proposition 4.5.
When the diffusion component exists, [38, Proposition 5] implies that $\hat{\sigma}(t)\to\sigma$ as $t\to 0$. In particular, $\hat{\sigma}_t t^{1/2}\to 0$ as $t\to 0$ and, thus, (A.22) above still holds. Let $\tilde{\sigma}(t):=\hat{\sigma}_t-\sigma$, then $\tilde{\sigma}(t)\to 0$ as $t\to 0$, and (A.22) can be written as

$$C_{BS}(t,\hat{\sigma}(t)) = \frac{\sigma}{\sqrt{2\pi}}t^{1/2}+\frac{\tilde{\sigma}(t)}{\sqrt{2\pi}}t^{1/2}-\frac{\hat{\sigma}(t)^3}{24\sqrt{2\pi}}t^{3/2}+O\left(\left(\hat{\sigma}(t)t^{1/2}\right)^5\right) = \frac{\sigma}{\sqrt{2\pi}}t^{1/2}+\frac{\tilde{\sigma}(t)}{\sqrt{2\pi}}t^{1/2}+O(t^{3/2}). \tag{B.37}$$

By comparing (4.4)-(4.5) and (B.37), we have

$$\frac{C2^{1-Y}\sigma^{1-Y}}{Y(Y-1)\sqrt{\pi}}\Gamma\left(1-\frac{Y}{2}\right)t^{\frac{3-Y}{2}} \sim \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}}t^{1/2}, \qquad t\to 0,$$

and, therefore,

$$\tilde{\sigma}(t) \sim \frac{C2^{\frac{3}{2}-Y}\sigma^{1-Y}}{Y(Y-1)}\Gamma\left(1-\frac{Y}{2}\right)t^{1-\frac{Y}{2}}, \qquad t\to 0.$$

The proof is now complete. □

C Additional proofs

Verification of (2.8).

By the very definition of $\mathbb{P}^*$ and (2.2), we have

$$\begin{aligned}\mathbb{E}^*(e^{iuX_t}) = \mathbb{E}(e^{(iu+1)X_t}) = \phi_t(u-i) &= e^{t\,C\Gamma(-Y)\left((M-i(u-i))^Y+(G+i(u-i))^Y-M^Y-G^Y\right)+ic(u-i)t-\frac{\sigma^2}{2}(u-i)^2t}\\ &= e^{t\,C\Gamma(-Y)\left((M^*-iu))^Y+(G^*+iu)^Y-M^Y-G^Y\right)+i(c+\sigma^2)ut+ct-\frac{\sigma^2u^2}{2}t+\frac{\sigma^2}{2}t}\end{aligned}$$

with $M^*=M-1$ and $G^*=G+1$. Next, using (2.3), we clearly have

$$tC\Gamma(-Y)(-M^Y-G^Y)+ct+t\sigma^2/2 = -tC\Gamma(-Y)\left((M-1)^Y-(G+1)^Y\right),$$

and thus,

$$\mathbb{E}^*(e^{iuX_t}) = e^{t\,C\Gamma(-Y)\left((M^*-iu)^Y+(G^*+iu)^Y-M^{*Y}-G^{*Y}\right)+ic^*ut-\frac{\sigma^2u^2}{2}t},$$

with $c^* = c + \sigma^2$. $\square$

Verification of (2.10).

Using (2.9),

$$\widetilde{\mathbb{E}}X_1 = \tilde{b} + \int_{\{|x|>1\}} x\tilde{\nu}(dx) = b^* + \int_{|x|\le 1} x(\tilde{\nu} - \nu^*)(dx) + \int_{|x|>1} x\tilde{\nu}(dx)$$
$$= c^* + \int_{\mathbb{R}} x(\tilde{\nu} - \nu^*)(dx) - CY\Gamma(-Y)((M^*)^{Y-1} - (G^*)^{Y-1}).$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}} x(\tilde{\nu} - \nu^*)(dx) &= \int_{\mathbb{R}} x(e^{\varphi(x)} - 1)\nu^*(dx) \\
&= \int_0^\infty x(e^{M^*x} - 1)\frac{Ce^{-M^*x}}{x^{1+Y}}dx + \int_{-\infty}^0 x(e^{-G^*x} - 1)\frac{Ce^{G^*x}}{|x|^{Y+1}}dx \\
&= \sum_{n=1}^\infty \frac{C(M^*)^n}{n!}\int_0^\infty x^{n-Y}e^{-M^*x}dx - \sum_{m=1}^\infty \frac{C(G^*)^m}{m!}\int_0^\infty x^{m-Y}e^{-G^*x}dx \\
&= C(M^*)^{Y-1}\sum_{n=1}^\infty \frac{\Gamma(n-Y+1)}{n!} - C(G^*)^{Y-1}\sum_{m=1}^\infty \frac{\Gamma(m-Y+1)}{m!} \\
&= -C(M^*)^{Y-1}\Gamma(1-Y) + C(G^*)^{Y-1}\Gamma(1-Y).
\end{aligned}$$

Hence, $\widetilde{\mathbb{E}}X_1 = c^*$ and (2.10) follows. $\square$

Proof of Lemma 2.1.

By (2.22), for any $0 < t \le 1$ and $v > 0$, we have

$$\begin{aligned}
\frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \ge t^{-1/Y}v\right) &= \frac{1}{t}\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \ge t^{-1/Y}v\right)\left(\mathbf{1}_{\{t^{-1/Y}v\ge N\}} + \mathbf{1}_{\{t^{-1/Y}v<N\}}\right) \\
&\le \frac{1}{t}\left(\frac{2C}{Y}tv^{-Y}\mathbf{1}_{\{vt^{-1/Y}\ge N\}} + \frac{N^Y}{(t^{-1/Y}v)^Y}\mathbf{1}_{\{t^{-1/Y}v<N\}}\right) \\
&\le \left(2CY^{-1} + N^Y\right)v^{-Y}.
\end{aligned}$$

The result follows by setting $\kappa = 2CY^{-1} + N^Y$. $\square$

Verification of (A.12).

Note that

$$t^{-1}\widetilde{\mathbb{P}}\Big(Z_1^+ + \widetilde{U}_1 \ge t^{-1/Y}u\Big) \le t^{-1}\widetilde{\mathbb{P}}\left(Z_1 \ge \frac{t^{-1/Y}u}{2}\right) + t^{-1}\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \ge \frac{t^{-1/Y}u}{4M^*}\right) + t^{-1}\widetilde{\mathbb{P}}\left(-\bar{U}_1^- \ge \frac{t^{-1/Y}u}{4G^*}\right).$$

Using Lemma 2.1, there exists a constant $0 < \kappa < \infty$ such that

$$t^{-1}\widetilde{\mathbb{P}}\left(\bar{U}_1^+ \ge \frac{t^{-1/Y}u}{4M^*}\right) + t^{-1}\widetilde{\mathbb{P}}\left(-\bar{U}_1^- \ge \frac{t^{-1/Y}u}{4G^*}\right) \le 2\kappa(4M^* + 4G^*)^Y u^{-Y},$$

for all $u > 0$ and $0 < t \le 1$. Clearly, (2.23) implies that, for any $0 < t \le 1$ and $u > 0$,

$$\frac{1}{t}\widetilde{\mathbb{P}}\left(Z_1 \ge \frac{u}{2t^{\frac{1}{Y}}}\right) \le 2^{2Y+1}\kappa u^{-Y} \le 32\kappa u^{-Y},$$

and (A.12) follows. $\square$

Verification of (B.7)

In light of (2.24), there exist $R > 0$ and $H > 0$, such that for any $u \geq H$,

$$p_Z(u) \leq Ru^{-Y-1}. \tag{C.1}$$

Now for $J_{12}(t,y)$, using (C.1), for any $0 < t \leq 1$ and $y > 0$,

$$\begin{aligned}
t^{\frac{Y}{2}-1}J_{12}(t,y) &= t^{\frac{Y}{2}-1}t^{\frac{1}{Y}-\frac{1}{2}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}y}^{\infty} up_Z(u)du \\
&= t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)}\mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}y \geq H\}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}y}^{\infty} up_Z(u)du \\
&\quad + t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)}\mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}y < H\}}\left(\int_H^{\infty} up_Z(u)du + \int_{t^{\frac{1}{2}-\frac{1}{Y}}y}^{H} up_Z(u)du\right) \\
&\leq t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)}\mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}y \geq H\}}\int_{t^{\frac{1}{2}-\frac{1}{Y}}y}^{\infty} Ru^{-Y}du \\
&\quad + t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)}\mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}y < H\}}\left(\int_H^{\infty} Ru^{-Y}du + H\widetilde{\mathbb{P}}(Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}}y)\right) \\
&\leq t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)}\int_{t^{\frac{1}{2}-\frac{1}{Y}}y}^{\infty} Ru^{-Y}du + t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)}\mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}y < H\}}H\left(\frac{H}{t^{\frac{1}{2}-\frac{1}{Y}}y}\right)^{Y-1} \\
&\leq y^{1-Y}\left(\frac{R}{Y-1} + H^Y\right),
\end{aligned}$$

and (B.7) follows. □

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